STUDENT-GENERATED EXAMPLES IN THE LEARNING OF MATHEMATICS

ANNE WATSON, UNIVERSITY OF OXFORD & JOHN MASON, OPEN UNIVERSITY

ABSTRACT

Examples, illustrations, occurrences and instances play a central role in the learning of mathematics. Deliberately getting students to generate examples is a particularly powerful tool in teaching, but while some of the possibilities they offer are used by many teachers, the full potential is rarely exploited. The authors came to this notion through reflection on their experience of observing teachers in classrooms, of getting students to take initiatives, and of creating examples while working on their own mathematics.

They examine the roles played by examples constructed and generated by students, give illustrations and analyses of the use of this tool, and develop a theory for the act of exemplification as an act of cognition. A framework is developed for researchers and practitioners to further this work.

EXECUTIVE SUMMARY

We use 'example' to mean a representative of any class of mathematical objects or actions, including techniques, concepts, properties, sets and so on. We describe how our own mathematics, and responses to some earlier work on mathematical questioning, led us to appreciate the importance of examples in mathematical learning.

We see the process of learning mathematics as coming into contact with phenomena concerning objects, techniques, relationships and structures known as mathematics, then extending awareness of the possibilities suggested by these, eventually generalising about facts or invariants. Familiarity with generic or informative examples which offer the range of possibilities is part of the process of making sense of mathematics (Polya 1962, Michener 1978).

The Cognitive Role of Examples

Examples can be used as *startup examples*, *reference examples*, *model examples*, and *counter-examples* (Michener, 1978). Are students always aware that examples represent or generate a larger class, and is it clear what is being offered which is generic rather than specific? Students may only use examples as templates to be followed using, for instance, different numbers, rather than demonstrations of how to manipulate representations or how to synthesise facts to achieve solution. They are often left to work out for themselves what is being exemplified and may end up focusing on irrelevant details. Relating an example to the generality it exemplifies is non-trivial.

Examples are not only used for assessment and motivation, however. Teachers frequently use examples in order to demonstrate and communicate the essence of mathematical concepts and techniques. Concepts are often communicated to students through the use of examples but leaving students to infer for themselves the point of an example, or a set of examples, can be ineffective because students may not know what is relevant. Students need to have plenty of examples because availability helps them overcome three problems: interpretation, knowing what to do and generalisation. Single examples may contain distractions.

Counter-examples can function as a device to disprove a conjectured formula, or to explain. Our searches for such examples have led us to develop the notion of *boundary examples*: those which are only just, or not quite, examples of the concept, idea or technique being learnt.

What has been written about SGEs?

SGEs can be used for assessment. Students' problem-posing and test-setting (generating examples of questions) is often used as both an assessment and self-assessment tool. Shifts in responsibility for test construction from teachers to students necessarily involves some kind of review of material. SGEs are also used to motivate and involve students.

Students attempts to create examples according to specified features can be used as starting examples for work on new concepts, and for transforming and developing knowledge of aspects of mathematics already familiar in different, limited or simpler forms.

Only a few writers have been more concerned with using SGEs to develop concepts than for assessment or motivational purposes. Concept development through exemplification need not be incidental or unexpected; it can be an explicit pedagogic aim. There is some evidence that students who consistently employ example generation as an integral part of their learning strategy, even when not specifically prompted to do so, undergo more shifts of concept image, give better explanations, develop broader example-spaces and have a more complete understanding of the taught concept.

Students find it hard to produce counter-examples. Explicit induction into the *mathematical* uses of exemplification and counter-exemplification are needed.

A framework of SGEs for concept development

There is very little written about use of SGEs for conceptual development in mathematics. We collected descriptions of practices observed in classrooms to identify conceptual re-organisation which would have to take place in order for students to respond to the tasks. We give five examples to illustrate five different kinds of experience students can have through generating their own examples during ordinary mathematical work:

- Experiencing structure
- Experiencing and extending the range of variation
- Experiencing generality
- Experiencing the constraints and meanings of conventions
- Extending example-spaces and exploring boundaries

We suggest that creating conditions in which students generate their own examples prompts students to learn actively, take initiative, acting upon ideas rather than be passive. Future work will include examining the psychology of constructing examples for oneself, and testing the efficacy of our framework for making sense of the different ways in which getting students to construct their own examples assists in their learning and subsequent performance.

INTRODUCTION

Two experiences in mathematical learning have influenced this work. For one of us, being told in some course notes to construct examples when reading theorems was a moment of revelation; until then it had been assumed that the provision of appropriate examples would always be done by the teachers. For the other, a lecturer had set the task of constructing examples to show that a sequence of increasingly constraining definitions was, in fact, a collection of distinct conditions on a topological space. These instances of being challenged to produce our own examples, rather than use those given by authors, had been crucial in our learning.

Prompted by responses to our earlier work on mathematical questioning (Watson & Mason 1998) and by the memories described above, we looked for instances of the use of student-generated examples (SGEs) amongst teachers and in literature, and began to identify tasks which would lead students to generate examples.

What is an example?

We take a broad view of what constitutes an example. Sowder (1980) used 'examples' to describe illustrations of concepts and also manifestations of principles. Teachers are used to providing ‘worked examples’ to augment those that are in textbooks. In this paper we go further by including mathematical objects constructed to exemplify properties; techniques; conditions; the uses of a theorem; and applications. Thus ‘example’ includes anything used as raw material for intuiting relationships and inductive reasoning: illustrations of concepts and principles; contexts which illustrate or motivate a particular topic in mathematics; and particular solutions where several are possible. Indeed we use *exemplification* to describe any situation in which something specific is being offered to represent a general class with which the student is expected to become familiar. A key feature of examples is that they are chosen from a range of possibilities.

We start from the position that examples and exemplification play a central role in learning mathematics. We summarise briefly some of the literature on uses of student-generated examples for assessment and motivation, and extend from this to focus on student-exemplification as a cognitive act. We then describe how we studied the issue and summarise a framework which offers the possibility for structuring further investigation.

THEORIES OF LEARNING

We see the process of learning mathematics as coming into contact with phenomena involving objects, techniques, relationships and structures known as mathematics, then extending awareness of the possibilities suggested by these, eventually generalising about facts or invariants (such as: the angles of all triangles on the Euclidean plane add up to 180 degrees; translating a shape preserves lengths and angles) and finally abstracting concepts which might then become objects for further levels of generalisation and abstraction (Tall 1991; Sfard 1991, 1994). Familiarity with generic or informative examples which offer the range of possibilities is part of the process of making sense of mathematics (Polya 1962, Michener 1978). Any learning theory which purports to apply to mathematics should include the potential for students to construct generalities from examples, whether the mechanism is seen as discursive adaptation, in which teachers and students might interact until they seem to be applying the same labels to the same sets of examples; enculturation of social practices, in which learners gradually absorb what is meant by the concept-label by experiencing more and more examples to which that label is applied; or psychological, in which learners compare new examples to their existing knowledge structures and adapt their understanding to take the new experience into account. Thus we see exemplification as having a central role in learning mathematics whatever the background learning theory.

THE COGNITIVE ROLE OF EXAMPLES.

Teachers frequently use examples in order to demonstrate and communicate the essence of mathematical concepts and techniques (Tall & Vinner 1981). For example, a particular subtraction problem is worked to illustrate 'subtraction'; a particular, if vague, sketch of a function will be used for work on continuity. However, students may see specific examples as yet more things to be absorbed, rather than as generic examples illustrating the significance of various conditions or properties. Students may use worked examples in textbooks merely as templates to be followed using, for instance, different numbers, rather than as demonstrations of how to manipulate representations or how to synthesise facts to achieve solution (Anthony 1994). For example, when a teacher is trying to teach students that elimination and substitution are good ways of resolving simultaneous linear equations, the students may be lost in the detail of "multiplying and subtracting" specific numbers.

Michener (1978), in looking at the roles played by examples of mathematical objects for mathematicians, elaborated distinctions between

> *startup examples*, used to arouse interest and to suggest how the theory would develop;
>
> *reference examples* which are learnt and used to test future conjectures or to revisit concepts;
>
> *model examples*, which are paradigmatic and generic (see also Mason and Pimm 1984);
>
> *counter-examples*, which demonstrate the boundaries of concepts or techniques, or which contribute to counteracting standard misconceptions.

She advocated enculturating students into asking themselves meta-cognitive questions based on these different roles of examples. She seems to take it for granted that most examples are already extant in the culture and so will come to students from authorities, by whom we mean teachers, textbook and examination writers. Although this is true for radically surprising examples, it is certainly not necessary for all examples to come from external authority.

There is a contrast between a single example seen as generic by the expert and a single example seen as something to learn by a student. For instance, a teacher might multiply some numbers by 0.3 to show, generically, that multipliers between 0 and 1 make other positive numbers smaller. Are students aware that this is an example of a range of numbers which act similarly? A teacher might offer $f(x) = |x|$ as a function which is continuous but not everywhere differentiable; is the student aware that this is just one of a class, indeed that it can be used to generate a large class having the same property? If students focus on specific examples, they may include irrelevant special features in their concept image because they did not know what was being exemplified (Charles 1980; Fischbein 1993). Therefore it is important for students to know about several examples, so that they can identify what is common to all of them (Wood, Bruner & Ross 1978; Reimann & Schult 1996; Bills 1996). Bills and Rowland (1999) consider the provision of examples as an invitation to the student to induce a generalisation, possibly intuitively, and while this may be the intention of most authors and teachers, there is a gap between intention and effect.

Might non-examples and counter-examples have a role to play? Sowder (1980) concluded that the inclusion of non-examples has an unpredictable cognitive role. Charles (1980) suggests that 'one conjecture worthy of investigation is that non-examples are more instructive for learning difficult concepts, whereas examples are more instructive for learning 'easy' concepts.' (p.19). Zaslavsky and Ron suggest 'students often feel that a counter-example is an exception that does not really refute the statement in question' (1998, p. 4-231).

There is a special kind of counter-example, referred to by Askew and Wiliam: 'The ideal examples to use in teaching are those that are *only just* examples, and the ideal non-examples are those that are *very nearly* examples' (1995, p.iii). Our searches for such examples have led us to develop the more general notion of *boundary examples* (Watson & Mason, 1998). For example: a general straight line may be given as $y = mx + c$ but this formulation may exclude lines such as $x = a$ from the student's experience. Because they are not encompassed by this general form, teachers find themselves having to introduce them deliberately in some way, as something extra to remember.

WHAT HAS BEEN WRITTEN ABOUT SGES?

It is almost always the responsibility of authorities to produce examples, and of students' to make sense of them. The usual deviations from this pattern are for assessment (students produce examples so that assessors can make sense of their mathematics) or for motivation (students pose problems or supply data and gain a sense of involvement and purpose). Most of the literature about SGEs is about these roles.

Many writers report on the positive effects of getting students to create test questions, which ~~are~~ often turn out to be harder but more motivating than those set by teachers (Bell and Swan 1995; Scriven 1995; Odafe 1998). Another kind of assessment through SGEs was studied by Waywood (1992) who asked students to use examples to illustrate what they had learnt. He detected a range of responses, from the inclusion of examples copied from other sources, through the use of examples to demonstrate an application or use or illustration, to the construction of annotated examples to summarise aspects of a topic or idea. He found that use of examples improved over time; students exemplified with increasing sophistication and fluency. In a similar way, the prompt "Show me an example of …" is used in self-assessment materials produced by Bell *et al.*, (1996). Of course, the more familiar learners are with being asked for examples, the easier they are likely to find it if suddenly asked for examples during assessment.

There have been several studies of the kinds of problems students pose, the motivating effects of working on these, and the complexities they introduce (Ellerton 1986 and 1988; Cudmore and English 1988; Van den Heuvel-Panhuizen *et al.* 1995). If helped explicitly, students can create problems which are structurally more complex than those offered by their teachers (English 1998; Silver et al., 1996; Silver and Cai 1996). Experience of problem-posing helps students to 'reason by analogy' when presented with similar questions (English, 1999).

Reasoning by analogy, raising mathematically interesting questions, and focusing on structure, are all features of mathematical thought which contribute to conceptual understanding. Asking students to pose their own problems and to illustrate their learning with examples might, with support, be expected to have positive cognitive effects.

USE OF SGES FOR CONCEPT DEVELOPMENT AND OTHER ASPECTS OF MATHEMATICAL LEARNING

Given the centrality of examples in the conceptual development of mathematical understanding, it is strange that so few writers recognise a role for SGEs in the cognitive aspects of learning mathematics, beyond assessment and the obvious affective aspect of being involved in creativity and in choice of task.

In UK classrooms, learners are often given a context which generates a sequence of numbers, and asked to 'find a formula'. Hewitt (1992) points out that such tasks can easily become mechanistic, but if the source of a sequence is questioned at a structural level, then SGEs can provide starting points for substantial mathematical work. For instance, students can be asked to identify special examples from their results, and discuss what it is that makes them special. Sowder (1980) reports using the prompt 'Give me an example, if possible, of …….', with the teacher taking responsibility for guiding students towards peculiar examples. SGEs can therefore be used as starting examples for work on new concepts, rather than merely for revisiting familiar ones.

Sadovsky (1999) challenged her students to exemplify division operations which give a dividend of 32 and a remainder of 27. She asked 'How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions can be found' (p.4 - 147). One outcome of her study was her conclusion that '… these problems are simultaneously a chance to find the limits of the arithmetic practices and enrich the conception of Euclidean division'.

But concept development through exemplification need not be incidental or unexpected; it can be an explicit pedagogic aim. Zaslavsky (1995) describes an effective use of generating student examples. She asked them to 'find an equation of a straight line that has two intersection points with the parabola $y = x^2 + 4x + 5$.' Attention was thus directed to features of a parabola and a straight line, rather than to algebraic or trial-and-error techniques for finding intersections. Each strategy led to further questions such as 'find an equation of a straight line which does not intersect twice with the parabola….'. Students encountered most of the analytical geometric syllabus, engaging with the structures and equations of straight lines and parabolae through their own examples.

A particularly interesting account of the use of exemplification for conceptual understanding is given by Dahlberg and Housman (1997). Working with tertiary students, they developed the notion of a "learning event" which they know has occurred when a student employs or communicates a new understanding of a concept. They frequently asked their students to give examples and counter-examples of concepts, alongside their own explanations, of definitions. They found that students who consistently employed example generation as an integral part of their learning strategy, even when not specifically prompted to do so, underwent more shifts of concept image, could give better explanations, developed broader example-spaces and hence had a more complete understanding of the taught concept. They claim that "example usage, particularly example generation and verification, is crucial for understanding a new concept." (ibid., p.284) .

Tirosh *et al.* (1991) point out how hard it is to produce a counter-example which may challenge previous belief. Zaslavsky and Ron (1998, p. 231) report that several of their subjects also found this difficult, trying first to 'spoil' the properties of elementary functions and only then trying to extend their example-space. The production of an example or counter-example and an understanding of its role may cause separate difficulties. O'Connor (1998) describes a child who was unable to produce mathematical counter-examples, but used counter-examples naturally in general conversation. She suggests that the discourse of exemplification and counter-exemplification is available to the child outside the mathematics classroom, but that its use in mathematics is limited by experience. With explicit induction into the *mathematical* uses of exemplification and counter-exemplification, inability to exemplify or counter-exemplify might *then* indicate lack of familiarity with the mathematics involved.

RESEARCHING THE USE OF SGES FOR CONCEPT DEVELOPMENT

During our reading and thinking about the cognitive role of examples we formed the hypothesis that SGEs had a far more important and exciting role to play in learning mathematics than had generally been recognised in the literature (with only a few exceptions). We set out to study the use of student-generated examples to promote conceptual understanding, consolidation and re-organisation of knowledge structures. As a result, we developed a framework of five uses for asking learners to generate examples for themselves. This framework does not include the assessment and motivation purposes which are already well-documented. Instead we focus on how mathematical meaning is constructed during, and because of, example-construction. The five types are:

- Experiencing structure
- Experiencing and extending the range of variation
- Experiencing generality
- Experiencing the constraints and meanings of conventions
- Extending example-spaces and exploring boundaries

Method

The method we used was a combination of theoretical and empirical approaches, of practitioner-research and observer-research. Like Maslow (1979) we are interested in the range of possibilities, though obviously we are stimulated and informed by descriptions and observations of practices.

Observations

We collected descriptions of practices observed in classrooms, the sampling being highly opportunistic. Over one year we made field-notes of suitable incidents in many classrooms during the course of our normal work; hence the data is grounded in the normal practices of experienced and novice teachers, in various phases and with various background qualifications.

The observations are taken from naturalistic settings, as far as is ever possible given that outsiders are present, and the incidents we noted were those in which the teacher asked pupils to contribute in ways which are more usually the responsibility of teachers, textbooks and test-writers. We noted how this was done and observed student responses where possible.

Our own pursuit of mathematics was a further source of data. We became increasingly aware of our use of exemplification when working on mathematics on our own or with others, just as we became more aware of it in every other mathematical arena.

Analysis

We then distilled these accounts, observations, and reports through a process of analysis, interpretation, and theme-seeking, in order to discern the conceptual re-organisation which would have to take place in order for students to respond to the tasks. At first, such identifications arose from our views of the nature and learning of mathematics. Some generic types of prompt were identified, distinguished by the potential they offered for students to think mathematically through responding to them. These constituted a draft framework within which to examine further data and our own mathematical experiences. A cyclical process of framework development took place which was hermeneutic in nature, our selection and analysis of data being influenced by the emerging frame, which was itself dependent on our interpretation of the data.

Alongside analysis and data collection we developed tasks arising from our grounded theorising which we could use with various audiences. For example, we suspected from our own mathematics that boundary examples play an important role. However we did not see boundary example construction in our observations, so we devised our own generic questions to prompt students to explore what is, or is not, included in various definitions, classifications and so on (Watson & Mason 1998). Our use of audiences was entirely ~~also~~ opportunistic and related to our normal work. The tasks we devised were used in naturalistic settings with our usual students and audience. Audience and learner responses were taken into account both for development of the framework and for further task refinement and development.

The data generation and collection methods we used are not easily systematised, being very dependent on our working lives, but the analysis and framework development which emerge are highly systematic and inherently self-validating. Exemplification was our focus in our multiple roles as researchers and practitioners in mathematics and mathematics education, and ideas emerging in one area of our work would be tested out in another, theoretical ideas being tested in practice, empirical observations being matched against theory, and learner response being a feature of all these facets.

Generative and illustrative data

Here are five illustrative accounts of different uses of SGEs for concept development. In each case, we demonstrate a generic strategy which might be applied in a variety of mathematical situations. Each account is labelled with one of the uses which we identified, but the reader will notice that, as in any teaching-learning situation, there are multiple ways to see the data. For instance, Account 1 is given here as an illustration of "experiencing structure" but if we had focused on the range of exercises produced it could also be seen as "experiencing range of variation".

Account 1 Experiencing structure

DM asked students to create examples of practice exercises for the whole class. The lesson was about solving linear equations with one unknown appearing on both sides of the equation. Students were asked to state a value for ~~of~~ x and, by repeatedly using the rule 'do the same to both sides', build up a complicated equation of the desired kind. For instance, one learner wrote: $x = 4$, $x + 1 = 5$, $x + 2x + 1 = 5 + 2x$, $3x + 1 = 5 + 2x$ and presented the final equation as her contribution.

The activity created an enthusiastic working atmosphere, and most of the class were able to solve the equations correctly, either during the lesson or later for homework.

Comments

The intention for their learning was that, by going through the process of building up the equation, they would have a better sense of how to solve (deconstruct) such equations. That is, they would experience the 'rules' for solving equations constructively in the context of making increasingly complex statements themselves. Students used four operations with single-digit numbers to produce structural escalation of complexity. In the problem-posing literature it was found that students could not produce structural complexity without explicit training, but here it happens because of the nature of the task.

Using SGEs this way is not primarily about motivation or assessment; the focus was on the students' experience of choice, complexity and construction. They become aware that to return to the value of x they have to 'undo' what they have done. Faced in future with linear equations ~~in future~~ they are more likely to recognise them as containing a value for x, and approach them as if undoing someone else's puzzle. We have called this strategy "burying the bone": concealing the essence of a mathematical situation (in this case, the value of x) and thus experiencing a mathematical structure. Such a strategy could be used wherever students are expected to learn a technique which involves simplification or transformation of a structure.

Account 2 : Experiencing and extending the range of variation

JF was working with 11 year-olds on multiplication. She had been explicit about the use of the distributive law to deal with different place values. Students were asked to contribute examples of multi-digit multiplications and show the class how they would calculate them. Some used the traditional approach of dealing with separate digits, such as 37 x 9 = 30 x 9 + 7 x 9, but others used *ad hoc* decompositions which suited the specific numbers, such as 37 x 9 = 40 x 9 – 3 x 9. A repetition of the exercise two weeks later showed a marked increase in the students' use of flexible approaches based on characteristics of the numbers involved.

Comment

By seeing each others' ideas students learnt about the range of possibilities. We assume that those who became more flexible were able to appreciate that some ways of decomposing numbers lead to easier or more interesting calculations, adapting their own approaches in the light of what they had heard. The development of exemplification was not explicitly forced by the teacher but happened through public valuing of a range of examples. In our first story students produced more complex examples in response to being urged to complexify their work. In this case some of the decompositions led to easier calculations than others and this may have appealed to some students; we are not convinced that sharing examples *inevitably* leads to students adopting a variety of techniques, but it seemed to do so in this case. We are not claiming that these students had therefore learnt the distributive law, but that exposure to a variety of possibilities, through use of SGEs, had given students access to a wider range of examples of the distributive law in action than is offered in textbooks. Such a strategy could be used wherever students may have a limited view of what is possible.

Account 3: Experiencing generality

A teacher, JM, asked students to write down:

a particular number leaving a remainder of 1 upon dividing by 7;

a number leaving a remainder of 1 upon dividing by seven which is peculiar in some way;

a general form of a number leaving a remainder of 1 upon dividing by 7.

For the first request, the teacher asked students to say what they had chosen. The first few were smallish numbers, but later contributions were of larger numbers and negative numbers. Discussion ensued about whether –6 or –8 would have a remainder of 1. For the second request students contributed a range of answers such as 700001, 1 and so on. The third request followed easily from these contributions.

Comment

This pattern of exemplification, *particular–peculiar–general*, provides a structure for revealing to students a range of possibilities, for structuring those possibilities, and for expressing generality (Bills 1996) as well as getting a sense of number structure. The second request led to the emergence of a common form from a range of examples, from which it was easy to generalise. This generic approach could be used when teachers want students to become familiar with a wide range of possibilities, to generalise about a class of objects, or to understand something structural about a collection of objects.

Account 4: Experiencing the constraints and meanings of conventions

A teacher, JK, had drawn several sets of coordinate axes on the board and asked 12 year-old students to suggest representations of the function 'Output = Input + 3'. There was class discussion of various attempts to join values from axis to axis. Eventually one student drew a nest of rectangles in which each horizontal edge was three units more than the vertical edge

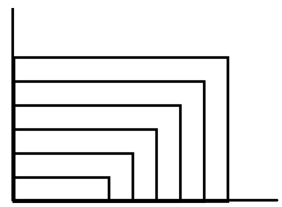

Figure 1

Another noticed that it would not be necessary to draw the whole rectangle, since dots for the vertex would be sufficient. Several students said this was 'like a graph'.

Comment

The exercise helped students connect number patterns to graphs, so that several ways of seeing such structures might be available to them in the future. The teacher was content to let students develop the representation for themselves so that they understood the principles behind mathematical representation in general. In the subsequent lesson, he pointed out that the convention was to use the horizontal axis to represent the input. We saw this as a use of SGEs of representations to help students experience the precision, clarity and purpose of conventions. If students have had to develop notations for themselves, and compared their usefulness, they are more likely understand and accept the strengths and idiosyncracies of conventional notations.

Account 5: Extending example-spaces and exploring boundaries

RT asked students working on classification of quadrilaterals to draw, in sequence, a quadrilateral; a quadrilateral with a pair of sides equal; a quadrilateral with a pair of sides equal and a pair of sides parallel; a quadrilateral with all these features and a pair of opposite angles equal. Most drew rectangles at first, but were forced to look at a wider range of quadrilaterals by a final instruction: check that the example given at any one stage will *not* do for the next stage. In other words, each further constraint narrows the range of possibilities.

Comment

It is limiting to have a rectangle come to mind when someone says 'quadrilateral' since this can lead students to make statements about whole classes which actually only apply to certain members. RT's intention was for students to see geometrical properties as constraints on freedom of choice, and also to encourage students to have a range of quadrilateral images available. This pattern of questioning conflicts with the tendency of students to offer rather special examples, so that in future students will have more examples to choose from and an extended notion of the concept.

DISCUSSION OF THE ROLE OF SGES IN LEARNING MATHEMATICS

We saw the above approaches as closely related to the work of Zaslavsky and her co-authors, Sadofsky, and the joint study by Dahlberg and Housman which we mentioned earlier. Each of our accounts addresses learning by involving students directly in construction activities. We saw more happening than mere problem-posing. Students were actively, noisily and verbally struggling with attempts to re-organise what they knew to fit the kind of example the teacher was seeking. Students were led away from limited perceptions of concepts and towards wider ranges of objects. They restructured their ways of seeing and experienced the creation of mathematical objects and notations.

Another notable aspect of these practices is the efficiency of the strategies in terms of teaching repertoire. All the accounts we have given demonstrate prompts which can be used in a variety of mathematical situations. A teacher needs only a few strategies to make SGEs a habitual expectation in the classroom.

The difference between following the form of someone else's examples and constructing one's own can be compared usefully to the difference between 'knowledge-telling' and 'knowledge-transforming' (Bereiter & Scardamalia 1987). For instance, if the student is asked to produce an example which is similar to one already available, then some generalising, at the level of identifying features or structures which may be common, may take place on the way to exemplification. If students are asked to produce examples which are structured slightly differently from the given one, knowledge-transformation *beyond* generalising a format is likely to take place.

All these arguments, and the related findings of Dahlberg and Housman (1997), lead us to conclude that the act of producing examples, particularly non-routine examples, is an act of learning, a cognitive act.

Students' experience of exemplification

We do not know for certain the students' psychological experiences of the request to generate their own material; the only evidence we have of success is their responses. From our experience as teachers we can say that it is likely that students, asked to think of a number, would give a small positive integer. If a teacher asks for a number between 0 and 1, with the intention of extending the range of numbers from which they normally choose, all we can know is whether, in that circumstance, students could respond. If they do respond, we still do not know whether they will think about fractions in the future. It is this teacherly experience of what is normal, the default mode of exemplification which mathematicians call "canonical" and Fischbein (1993) called the "figural concept", which has led to those with whom we have worked commenting on the power of some of the techniques we have collected, identified and classified.

Finally, from our experience as learners we can say that the process of searching for and constructing examples requires detailed, careful and creative work on the conditions required for examples, and appraisal and reappraisal of what is already believed to be possible.

In Table 1 we summarise these techniques as a framework for teachers and researchers to consider further. In each type we have described the intended experience of the pupil who engages with the task, and ways in which they might come to those experiences.

Experiencing structure	through executing and reversing processes: 'doing and undoing'; through exemplifying within constraints on variables; through complexifying a mathematical statement ('burying the bone')
Experiencing and extending the range of variation	through seeing others' examples; through using and developing representations; through being offered different constraints; through constructing different questions which give the same answer, different answers to the same question
Experiencing generality	through seeing a pattern in examples produced; through producing particular-peculiar-general examples; through testing a range of examples
Experiencing the constraints and meanings of conventions	through comparing to conventional methods their own mathematical creations; their own representations; their own suggestions of what might be useful
Extending example-spaces and exploring boundaries	through being asked to give an example of a certain type; through being asked to illustrate new concepts; through exemplifying what is and what is not, what can be and what cannot be; through meeting certain conditions but not meeting others; through being asked to exemplify what *cannot* be done within certain constraints

Table 1

FINAL COMMENTS

Generally, producing examples requires some kind of generalisation to take place, even if it is only generalisation of form. The act of creating an example is therefore an act of cognition, often with a positive affective component. We have presented five types of experience which students can have through being asked to generate examples. We have suggested that creating conditions in which students generate their own examples of all kinds prompts them to learn actively, to take initiative, acting upon ideas rather than be passive. Future work will include examining the psychology of constructing examples for oneself, and testing further the efficacy of our framework for making sense of the different ways in which getting students to construct their own examples assists in their learning and subsequent performance.

We thank all the teachers and colleagues whose teaching practices, or whose willingness to carry out tasks we offered, contributed to this work.

BIBLIOGRAPHY

Anthony, G. (1994). The role of the worked example in learning mathematics. In A. Jones et al. (Eds.), SAME papers (pp. 129-143), University of Waikato, Hamilton.

Askew, M. and Wiliam, D. (1995). *Recent Research in Mathematics Education 5-16.* London: HMSO.

Bell, A. and Swan, M. (1995). Learning how to learn. *Mathematics Teaching* 153, p14-17.

Bell, A., Swan, M., Shannon, A., and Crust, R. (1996). Pupils' awareness of their learning. *Mathematics Teaching* 154, 6 – 9.

Bereiter, C. & Scardamalia, M. (1987). *The Psychology of Written Composition.* New Jersey: Erlbaum.

Bills , L. (1996). The use of examples in the teaching and learning of mathematics. In L. Puig and A. Gutierrez (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp.2-81-2-88)Valencia, Spain.

Bills, L. & Rowland, T. (1999). Examples, Generalisation and Proof. In L. Brown (Ed.) *Making Meaning in Mathematics*: a collection of extended and refereed papers from the British Society for Research into Learning Mathematics, Visions of Mathematics 2, Advances in Mathematics Education 1 (pp.103-116) York: QED.

Charles, R.I. (1980). Exemplification and Characterization Moves in the Classroom Teaching of Geometry Concepts. *Journal for Research in Mathematics Education*, 11(1), 10 - 21.

Cudmore, D. and English, L. (1998). *Using intranets to foster statistical problem posing and critiquing in secondary mathematics classrooms.* Paper presented at the annual meeting of the American Educational Research Association, San Diego.

Dahlberg, R.P. and Housman, D.L. (1997) Facilitating learning events through example generation. *Educational Studies in Mathematics* 33. 283-299.

Ellerton, N. (1986). Children's made up mathematics problems: a new perspective on talented mathematicians. *Educational Studies in Mathematics* 17, 261-271.

Ellerton, N. (1988) Exploring children's perceptions of mathematics through letters and problems written by children. In A. Borbas (Ed.) *Proceedings of the 12th Conference of the International group for the Psychology of Mathematics Education* (pp.280-287). Veszprem, Hungary

English, L. (1998). Children’s Problem Posing within Formal and Informal Contexts. *Journal for Research in Mathematics Education,* 29(1) 83-106.

English, L. (1999). Reasoning by Analogy: a fundamental process in children’s mathematical learning. In L.V. Stiff and F.R. Curcio (Eds.) *Developing Mathematical Reasoning in Grades K-12* Reston, Va: National Council of Teachers of Mathematics.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics,* 24 (2) 139-162.

Hewitt, D. (1992). Train Spotters’ Paradise. *Mathematics Teaching* 140, 6-8.

Mason, J. & Pimm, D. (1984). Generic Examples: Seeing the General in the Particular. *Educational Studies in Mathematics* 15(3), 277-290.

Maslow, A. (1971). *The Farther Reaches of Human Nature*. New York: Viking Press.

Michener, E. (1978). Understanding Understanding Mathematics. *Cognitive Science,* 2 361-383.

Odafe, V. U. (1998). Students generating test items: a teaching and assessment strategy. *Mathematics Teacher*, 91(3), 198-202.

O’Connor, M.C. (1998). Language socialization in the mathematics classroom: discourse practices and mathematical thinking. In M. Lampert and M. Blonk (Eds.) *Talking Mathematics in School: studies of teaching and learning (*pp. 17-55) Cambridge: Cambridge University Press.

Polya, G. (1962). *Mathematical Discovery: on understanding, learning, and teaching problem solving*. New York: Wiley.

Reimann P. & Schult, T. 1996, Turning Examples Into Cases: acquiring knowledge structures for analogical problem-solving, *Educational Psychologist*, 31 (2) p123-140.

Sadovsky, P. (1999). Arithmetic and Algebraic Practi(c)es: possible bridge between them. In O. Zaslavsky (Ed.) *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, (pp. 4-145-4-152) Haifa, Israel.

Scriven, M. 1995, Student Involvement in Assessment, *Assessment Forum*, Vol V (3).

Sfard, A. 1991, On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. 1994, Reification as the Birth of Metaphor, *For the Learning of Mathematics*, 14 (1) p44-55.

Silver, E. A. and Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education,* 27(5), 521-539.

Silver, E.A., Mamona-Downs, J., Leung, S.S., Kenney, P.A. (1996). Posing mathematical problems. *Journal for Research in Mathematics Education,* 27(3), 293-309.

Sowder, L. (1980). Concept and Principle Learning. In R.J.Shumway (Ed.) *Research in Mathematics Education*, Reston, Virginia: NCTM.

Tall, D. (1991). *Advanced Mathematical Thinking.* Dordrecht: Reidel.

Tall, D. & Vinner, S., (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, 12(2), 151-169.

Tirosh, D., Hadass, R., Movshovitz-Hadar, N. (1991) Overcoming overgeneralizations, the case of commutativity and associativity. In F. Furinghetti (Ed.) *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education*, Vol III, (pp. 310-315) Assisi, Italy.

Van den Heuvel-Panhuizen, M., Middleton, J.A., and Streefland, L. (1995). Student-generated problems: easy and difficult problems on percentage. *For the Learning of Mathematics,* 15(3), 21-27.

Watson, A. & Mason, J. (1998). *Questions and Prompts for Mathematical Thinking.* Derby: Association of Teachers of Mathematics.

Waywood (1992) Journal writing and learning mathematics. *For the Learning of Mathematics* 12(2), 34-43.

Wood, P. Bruner, J. & Ross, G. (1976). The Role of Tutoring in Problem Solving. *Journal of Child Psychology and Psychiatry,* 17, 89-100.

Zaslavsky, O. (1995). Open-ended tasks as a trigger for mathematics teachers' professional development. *For the Learning of Mathematics,* 15(3), 15 - 20.

Zaslavsky, O. and Ron, G. (1998). Students' understandings of the role of counter-examples. In A. Olivier and K. Newstead (Eds.) *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, (pp. 4-225-4-233) University of Stellenbosch, South Africa.

AUTHORS:

Anne Watson

University of Oxford Department of Educational Studies

15 Norham Gardens

Oxford OX26 3ED

anne.watson@edstud.ox.ac.uk

John Mason

Centre for Mathematics Education

Open University

Milton Keynes MK7 6AA

j.h.mason@open.ac.uk