Low attainers exhibiting higher-order mathematical thinking

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In an earlier issue of *Support for Learning* I said that there is much to be written about how to support low achieving students in the development of mathematical thinking (Watson, 2001). In this paper I will give several examples of low attainers showing characteristics of mathematical thinking often associated with high achievers. These suggest that whatever is hampering their mathematical attainment is not an inability to think in the ways which enable others to succeed.

Recent brain research describes how natural brain functions and human behaviour form the foundations of mathematical thinking (for example, Butterworth,1999; Dehaene, 1997; Lakoff and Nunez, 2000). The implication is that abilities to think mathematically and understand mathematical concepts are adaptations of our natural brain and bodily activities, given elementary inborn propensities to recognise and compare small numbers. If this is true, then those who are better at mathematics than others must be better at learning from mathematical experiences, rather than have innate superior abilities.

The national emphasis in the UK on raising standards has led to higher achievement for some students, according to current measures. This shows that teaching *can* make a difference to achievement when guided by higher expectations. But the emphasis on raising the achievements of those who are slightly below the target standards ignores those who are well below those levels.

In this paper I use examples from classrooms to generate questions about the potential of currently low attainers to do better in mathematics.

Some clear barriers to learning

For example, the ability to choose appropriate and efficient strategies, and to adapt them if necessary, is seen as a characteristic of high-achieving mathematicians (Krutetskii, 1976). In Harding's (2000) detailed examination of students' take-up of mental arithmetic strategies she found that a few students, even after close one-to-one teaching, were unable to develop a repertoire of strategies. The strategy they used changed as the project succeeded, but they changed from *always* doing calculations one way to *always* doing them another. They seemed to find being offered alternatives confusing. These students appeared to be unable to choose strategies, in spite of careful personal teaching.

Another example of a clear barrier to learning is given by Tomkys (2001), who tells of a young student whose performance in subtraction calculations was correct until the end, where she always subtracted one from the units as a final operation. In spite of being able to spend several hours in one-to-one teaching, the teacher was unable to locate the source of this error, supposing in the end that it was a misapplication of a rule which might sometimes apply correctly to the 'tens' column. It is a common experience that some students become fixated on applying inappropriate rules. Even when the student successfully demonstrated subtraction with apparatus, and followed or even initiated correct chains of reasoning with the teacher, she still subtracted one after all other work had been done. It was several weeks before the student was able to adapt this incorrect strategy.

In the study I am about to describe there was one adolescent student out of eleven low attainers who showed no evidence at all of mathematical thinking processes. In a one-to-one interview he was asked to 'double' nine in three different contexts during half-an-hour. Each time he used a 'count all' strategy afresh; he was unable to recall that he had already done the calculation, or to recall whether the answer he reached was correct or not.

Above we have three examples of barriers to learning: inability to cope with choice; inability to give up a 'known' rule; and inability to remember very recent results, or to shift from counting to a cardinal concept of number (for similar cases, see Butterworth, 1999). Any teacher will have similar stories about students from whom they gain very detailed knowledge of specific barriers to learning, but it is also worth reporting that *most* students in Harding's study, rather than just the highest achievers, *did* exhibit the higher-order behaviour described by Krutetskii. It does seem to be true that explicit teaching of useful ways to think can improve the attainment of many. The CAME project, reported in an earlier *Support for Learning* (Adhami, 2001), and its predecessor in science teaching, the CASE project (Adey and Shayer, 1994), find that students who had been encouraged to think enquiringly in science or mathematics did better in a range of other unrelated subjects than a comparative group who had not had the same experiences. These studies lead me to ask: what ways of thinking are employed by successful mathematical learners and does it help to teach these explicitly to low attainers?

A proficiency approach to mathematical thinking

The questions I have posed so far assume a deficit model of the mathematical thinking skills of low attainers. However, Harries' work suggests a different approach (2001). He placed students in a situation in which they could leave a trace of their thinking, showing clearly how they completed a mathematical drawing task. Rather than consider teaching low attainers to act out the characteristics of successful mathematicians, we could continue Harries' exploration and look at proficiencies rather than deficiencies. What kinds of thinking, which might lead to higher attainment in mathematics, are manifested in the work of low attainers? To recognise some of these, and work for their enhancement, avoids the temptation to reduce complex ways of thinking to lists of superficialised heuristics which are then performed as algorithms, rather than experienced as deeply creative processes.

Harries' work, reported in an earlier issue of *Support for Learning* (2001), looked at how a sample of low attaining students had worked on some mathematics. Using Logo, he was able to keep records of the various ways in which they had tried to instruct the computer to draw and move simple objects, and then to use those objects as new elements in the creation of further shapes and patterns. He found evidence that most of them were able to transcend a purely manipulative approach through seeing objects they had made at one stage of their work as tools to use in the next stage. These types of shift characterise the genesis of mathematics, as generations of mathematicians developed the subject as we know it today. For example, addition develops from getting an answer by enumerating the combined contents of two sets, through an appreciation of cardinality, to using known number bonds as tools for harder addition sums.

This shift is inherent in the structure of mathematics, so it is also a useful shift for a learner to make.

That Harries found such a shift among some low attainers suggests that there might be other forms of mathematical thinking which have been characterised as features of the work of high achievers, but which some low attainers *can* do. For some reason these forms may not have been harnessed in the teaching they have received, or their response to it. I do not want here to discuss reasons why this harnessing has not happened, but to point to some other types of mathematical thinking which I have found some low attainers to be using, or to be able to use.

The study

The following examples come from research in a class of low-attaining Year 9 students in a city comprehensive school. The intake of the school was skewed towards lower-than-average attainment levels, and this particular class contained about eleven regular attenders (rather fewer than were on roll) with a variety of behavioural problems, language differences, patchy school histories and minor learning difficulties. There would usually be one or two support assistants in addition to the usual teacher. I had arranged to observe one double lesson each week with the intention of finding out how students would respond to particular kinds of questioning and prompting within their normal lessons. In the event, many factors typical of school life prevented the study proceeding as planned (seeWatson, 2000) and I became a teacher or support assistant as well as an observer and interviewer. Nevertheless, I was able to build up a substantial record of classroom incidents in which students showed that they could make shifts, either independently or with suitable prompts, which led them into forms of mathematical thought beyond the superficial features of the given task. Here I shall discuss two examples.

An example of flexible use of representation

Near the end of a lesson in which students had been working with fractions I drew some identical squares on the chalkboard and asked students to come to the board and indicate how to quarter the squares (figure 1). The first offering was a drawing of the obvious vertical and horizontal lines; the next was the two diagonals; the third, after a short wait, was a dissection using three vertical lines to give four congruent strips. So far students had interpreted the task as producing four congruent shapes and there had been a sense among those who came to the board of taking responsibility for getting the usual examples done. After waiting in silence a while I drew a version with two 'strip' quarters and two 'square' quarters. There was a pause while they considered what I had done. I asked them to vote on whether they believed I had cut the square into quarters or not. I was trying to encourage them to shift from seeing fractions as congruent shapes to seeing fractions as quantities, in this case making a link with area. Some of them were able to do this, offering area as the way to 'see' it. One student saw that further cutting and rearranging of the pieces would allow you to see that they were equal in area. After this I paused and waited to see if anyone would offer other ideas. One student offered the same dissection rotated through a right angle. Then Darren drew a version with three slanting lines, including one diagonal.

Because these students are labelled 'low attainers' it is tempting to assume that he was wrong; that he had not understood that the four pieces had to be equal in area. Giving up congruence as a criterion *may* have suggested that one could give up equality of any kind. He could also have been trying to extend the idea of cutting diagonally by mixing it with parallel cuts. This seems likely, given how the diagonal idea had not yet been developed further. I asked him if he was sure about where his lines should go. He said he was not, and that he did not know where to put them 'to make each half into halves'. Of course, my question 'are you sure?' may have given him the clue to answer 'no', but his voluntary elaboration of the negative answer revealed that he knew more: he had realised that deciding where to put the lines was hard. Another student, the one who had suggested rearranging pieces before, said 'you can cut them up and move them around' but could not show us how to do that. The end of the lesson came and I was unable to pursue this further.

It is hard to say exactly what was going on for individuals in this story. Most of them were able to shift from seeing fractions as congruent parts to fractions as something about equal areas, that is, they were able to use the spatial representation in a different way, as a result of being offered the idea and asked to work with it. A few had taken this further and made a definite link with area. One had been prepared to abandon the square and move shapes around, so that demonstrating equality was more important than retaining the original shape. No one had abandonned the notion of symmetry playing a role in this. It seemed as if most of the students were able to adapt their interpretations of a very familiar representation, but this happened to varying extents. Again, I was struck that low attainers were able to make such shifts, given the opportunity to do so.

Symbolisation can create difficulties for learners in mathematics. All truly mathematical ideas are necessarily abstract and we only have access to them through the way they are represented on the page or with manipulable materials. It follows, then, that learners whose understanding of an underlying concept can be independent of a particular representation of it are in a better position to extend their understanding outside the confines of one representation, or to recognise a familiar concept in unfamiliar clothes. Dreyfus (1991) therefore points to flexible use of representations as a useful advanced skill in working with mathematics.

An example of abstraction

Almira had been given a list of coordinates of points in the positive quadrant. The task was to plot them all on a coordinate grid and join them up in the order given to make a picture. The finished picture was thus an in-built self-checking device. When I looked at her work she had drawn in a number of unconnected lines, rather than systematically going from point to point in the order offered. On closer inspection it was clear that she had gone through the list of points and found all adjacent pairs which were of the form (a,b) with (a+1, b-1) and had entered those points and drawn the associated vector (1,-1). In other words, she had focused on the relationships between the points rather than the points themselves, and selected all those for which the relationship was the same. In drawing these in she had to use the actual positions of the points, but had created for herself a higher level task involving identifying relationships, classifying them, and performing all those of the same type at once. In conversation with her I found out that she had done this to make the task 'more interesting' and had thought it more efficient. In fact, she was having to do deeper thinking and make more passes through the data in order to complete the drawing, so could be said to be doing *more* work, but her interest was in the relationships and not the finished drawing. Here was a low attaining student who had voluntarily made the task more abstract and complex by shifting her focus from the coordinates to the relationships between objects, using coordinates as a tool to get her started, and using the drawn lines themselves conceptually. In a sense she had invented the concept of 'vector' (qualities which, in two dimensions, can be represented by lines which have a given length and direction) for herself. This shift would not be apparent in the finished drawing and without a record of how the drawing was constructed, such as Harries' students were able to produce, or my fortuitous observation, Almira's ability to apply higher order mathematical thinking to the task would be lost.

Almira demonstrated a deliberate shift from process to concept. To extend the earlier example, addition develops from being the process of combined enumeration to being seen as only one of many ways of combining two or more numbers, an example of the concept of binary operation. Such ability to combine the roles of process and concept has been described as a component of advanced mathematical thinking (Tall, 1991, p.254). Equating, for example, leads us to understand equations; then having equations as a tool lets us use them in complex situations and treat them as objects in their own right, not just as relationships between other objects. Thus 'equating' can be a technical skill, a concept, or an abstraction depending on how it is used and how it is seen by the learner. In Almira's work, she turns the process of joining dots into a vector concept, and then works with similarities and differences within this new-for-her concept.

Issues for teachers

Harries (2001) points to the work of Krutetskii (1976) who characterised the thinking of successful Russian mathematics students as a possible source of information on ways to think. Other sources of knowledge about mathematical thinking would be Polya (1962) and Schoenfeld (1985), who describe problem-solving heuristics; Mason et al (1982) talk more generally about ways to work with mathematices, both solving problems and exploring new ideas; Dubinsky (1991) discusses the Piagetian process of reflective abstraction as a component of advanced mathematical thinking. None of these writers suggest that explicit teaching can help others think in these ways, nevertheless the publication of their ideas suggest an implicit theoretical belief that identifying, naming and disseminating such descriptions makes them available for wider use. Teachers, however, often adopt different teaching styles for different teaching groups. This suggests a *professional* belief that it is not appropriate to expect lower achievers to approach topics in the same way that high achievers might. High achievers might be expected to grasp abstract concepts quickly, manipulate and explore them, deal with alternatives, find their way through complexities, and so on. Instead, lower achievers are seen to need more structured, step-by-step approaches, mathematics presented in practical and contextualised forms, and to practice simple algorithms.

But there is no standard recipe for mathematical success. Dahl (2000) found that successful year 13 students had a variety of different ways of working on new mathematics: there was not even commonality about whether they worked by accepting the general statements of others and manifesting them through specific examples, or whether they used specific examples to help them understand generalities for themselves. There were also various ways of using intuition and short-cuts in their work. If there is no universal recipe for mathematical thinking, and an extant belief that low attainers think in lower level ways, what can teachers do to discover and use higher-order thinking which might be taking place?

In Harries' work, a situation had been created in which traces of thinking could be 'read' by the teacher, but the activity itself structured the opportunity given to learners to make the shifts he sought. In Darren's case, the task had been chosen to give an opportunity to go beyond the usual responses; the teacher waited for a while and then gave a clue that other things were possible; creative responses were expected and, eventually, one was given. In Almira's case, the activity offered the opportunity to approach the task in a variety of ways, from mundane to abstract. Once her approach had been seen and recognised by a teacher who may not already have seen the possibilities, questions and prompts could be devised for other learners to help them make the same shift. In this case, I had proceeded to ask other students 'What is the same about these two lines? Can you see patterns in the numbers for these two lines?' and so on. By doing this, I found that nearly all the class could make a similar shift, given the prompting and the chance to do so.

So there are several questions which arise for teachers of low attainers in mathematics. Is it possible to structure work so that higher-order thinking is encouraged and noticed, even in simple mathematical situations? Can we trust more low attainers to think in mathematically-sophisticated ways, even in simple situations? Do we recognise higher-order approaches when they are being used, then value them by making them explicit for others?

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Figure 1

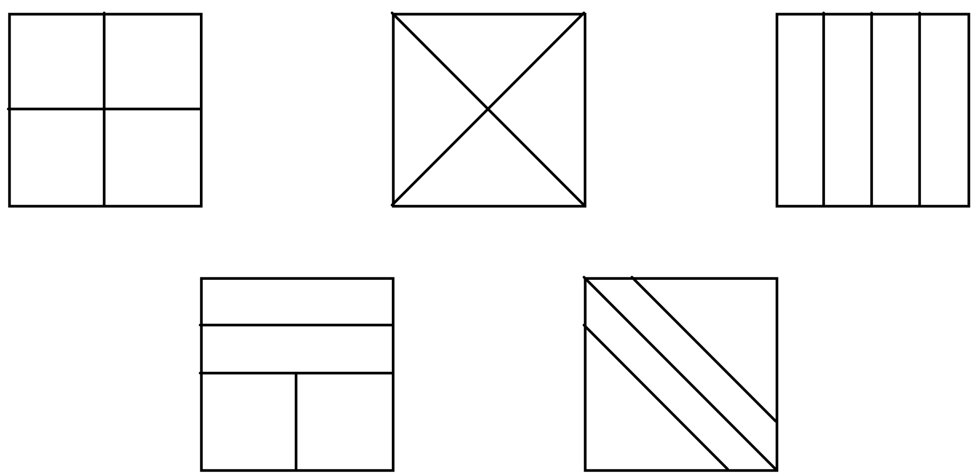
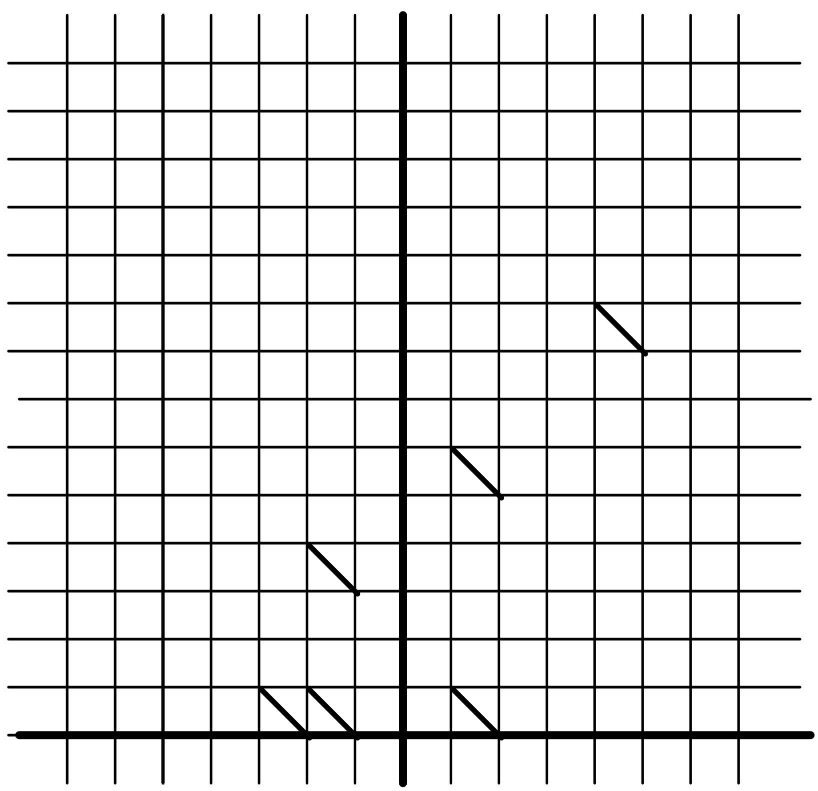


Figure 2



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