Teaching mathematical problem solving in primary school

Teacher Handbook

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A research briefing and introduction to the teaching programme

1. What is new about this problem solving programme for primary school

In the last three decades, researchers have reached a much more precise understanding of mathematical problem solving in primary school, particularly in the domain of quantitative reasoning. There is a better understanding of what characterises quantitative reasoning, the distinctions that are important in order to understand children’s reasoning, and the challenges for teaching children in ways that improve their problem solving ability in mathematics. This research briefing summarises some of these important ideas and describes a teaching programme that was found effective in research in improving children’s problem solving ability. We start by summarising the new ideas and then describe the problem solving programme that we implemented with children in our research project.

2. Mathematical problem solving and quantitative reasoning in primary school

One of the aims of teaching mathematics in primary school is to help children think about situations in mathematical terms. This aim is commonly referred to as teaching children “to mathematise” the world. In order to mathematise the world, children need to learn to represent the world mathematically and to reason logically about these representations.

This learning begins even before children start school. When children learn to count, for example, they are learning to use mathematical representations, numbers, for quantities
that exist in the world, such as the sweets the child has, the people in the child’s family, or the pencils inside the box that Granny bought. But knowing how to count and use numbers to represent quantities is only one part of mathematising the world: children also need to learn to think logically about quantities. Many children also start school with some knowledge of how to think logically about quantities and how to use mathematical signs to solve numerical problems about the world. For example, we can ask children at the time they start school, when they are between 5- and 6-years-old, about a simple situation such as “Karen had 4 pencils; her mother bought her 3 pencils; how many pencils does she have now?” If it helps, we can give them some blocks that they can use to represent the pencils. The vast majority of the children (usually more than 80%) will be able to answer this question correctly.

This may not seem remarkable but if we think about the children’s accomplishment we should actually marvel at their quantitative ability. First, we should admire their ability to create quantitative representations. If they take 4 blocks to represent the pencils that Karen had at first and 3 blocks to represent the pencils that her mother gave her, they are using a quantitative representation: the number of blocks matches exactly the number of pencils and therefore is a numerical representation of the pencils in the story. We should take this ability rather seriously because many children up to the age of 4 will not be able to do this: if asked to give 4 things to a doll, they do not take exactly 4 but simple take a bunch. They may know how to count but do not understand yet how to use counting to create an exact quantity, and therefore cannot create an exact quantitative representation. Second, we must admire their ability to manipulate quantitative representations to get to logical conclusions. If they count the first 4 blocks – one, two, three, four – and continue counting
the other 3 – five, six, seven – they have demonstrated an interesting bit of logic: they have shown that they understand that the sum is the union of the two parts. Once again, this understanding should not be taken for granted. Many children up to the age of four cannot tell that counting should be used differently if we want to compare two quantities, in which case we count each one separately, and if we want to know how many altogether in two sets, in which case we count the second set continuing to count from the number of objects in the first one. So, many children start school with some knowledge that they can represent quantities using numbers and can also use counting in different ways, depending on the logic of the situation, to arrive at answers to quantitative problems.

3. Some important distinctions for understanding quantitative reasoning in primary school

The example we have just given helps us to think about some of the basic distinctions that are important in quantitative reasoning because it shows that children may know the numbers (i.e. know how to count) and not know how to use them to represent quantities. The first distinction is between numbers and quantities. It is easy to see that numbers and quantities are not the same thing because we can reason logically about quantities without representing them with numbers. If you know that Dylan has more marbles than Alan and that Alan has more marbles than Sam, you know that Dylan has more marbles than Sam, although you have no idea how many marbles any of these boys has. This shows that one can reason logically about quantities without representing them with numbers.

There is a second distinction that is important in quantitative problem solving, which is the difference between quantities and relations between quantities. I could now tell you
that Dylan has 3 more marbles that Alan and that Alan has 2 more marbles than Sam; now you know that Dylan has 5 more marbles than Sam, and still you have no idea how many marbles each of the boys has. We have represented the relation between the quantities with numbers but have not represented any of the quantities with numbers. One can operate on the numerical representations of relations between quantities without knowing the quantities.

A third distinction that one must make in order to understand quantitative reasoning is the distinction between relations and operations. This distinction is also illustrated in the preceding example: one can add the two differences to find the third relation in the problem, the difference between the number of marbles that Dylan has and the number than Sam has. The operation of addition helps us to find out the third difference that can be identified in the situation. It is important to emphasise this distinction here because it would be easy to confuse subtraction, which is an operation, with difference, which is a relation between two quantities.

The distinctions between numbers, quantities, relations and operations are important because these are the units of thinking that children must learn to use when solving quantitative problems. Teachers need to be aware of these distinctions because quite often children’s difficulties in problem solving result from confusing quantities with relations or relations with operations.
3. Children’s difficulties in solving quantitative problems

Teachers are very aware that solving quantitative problems makes two kinds of demand: children need to be able to decide which operation to carry out and they need to know how to carry out the calculation. The decision about which operation to carry out is based on the children’s understanding of the relations between quantities in a problem. For this reason, this step in problem solving has been called relational calculation or mathematical reasoning. The second step, the operation with the numbers, is called numerical calculation.

Teachers are very aware of this distinction because they recognise that children may know how to carry out a calculation but not know when it is the right one to solve a problem. It is worthwhile noting that, if children know which operation they should carry out but do not know how to compute it, they can use a calculator. Indeed, many adults know, for example, that if they are told the area of a square, they can find out the length of its sides by calculating the square root. Many adults do not recall how to calculate the square root but they can solve the problem with a calculator. However, if they do not know which operation to carry out, a calculator is useless. Often children may be in the same situation, and having access to a calculator is no help to them.

The sorts of problem that children can solve when they start school require very simple mathematical reasoning: they know how to solve problems where two sets are put together and they are asked about the sum, or one set is taken away from a total, and they are asked about what was left. Problems that require more transformation of the information before choosing the operation (i.e. require more mathematical reasoning) are significantly more difficult. Given a story problem with an initial quantity, a change in this
quantity, and a result, one can create three types of problem by changing what we ask the children to find out: we may leave the initial quantity unknown and ask the children to figure this out, we may leave the change unknown, or we may leave the resulting quantity after the change unknown. For example, the problem “The postman had to deliver some letters. He delivers 7 to the houses in the first street. He still has to deliver 12 letters. How many letters did he have when he arrived in the first street?” is a start-unknown problem, because we know the change (he delivered 7 letters in the first street) and we know the result (he still has 12 letters). This problem requires some mathematical reasoning in order for the correct operation to be identified. The children must reason that, before the postman delivered 7 letters, he had 7 letters more than he has now, and therefore in order to find out the quantity at the start, they need to add 7 and 12.

We gave story problems to children in Year 2 (mean age 7 years and 1 month) and Year 3 (mean age 8 years) and asked them to enter in the calculator the operation that would allow them to find the answer. Thus, we wanted to know how well they would reason about the quantities and we were not concerned with the numerical calculation. We asked them to solve the three types of problem that we mentioned earlier on, start-unknown, change-unknown, and result-unknown problems. Result-unknown problems require little relational calculation: if the story is about increasing a quantity, the calculation required is an addition, and if the story is about decreasing a quantity, the calculation required is a subtraction. For this reason, they are called direct problems. The start-unknown and change-unknown problems require relational calculation: if the change was about an addition, we need to use a subtraction to find out the quantity at the start, and for this reason such problems are called inverse problems. The level of success in these problems,
out of 4 questions, for children in years 2 and 3 was 3.15 correct answers when the result was unknown, 0.61 correct responses when the change was unknown and 1.20 when the start was unknown. Thus, the children in years 2 and 3 did solve addition and subtraction calculation relatively well, as demonstrated by the level of success in the result unknown problems (79% correct answers) but they did not show much success in the problems that required reasoning about the inverse relation between addition and subtraction, as their level of success was well below 50%.

4. Two sorts of relational reasoning

One final distinction, which is subtle but nevertheless important, is a distinction between necessary relations, which are part of the logic of the situation, and relations that can be established by someone who is thinking about a problem, which are referred to as contextual relations. In a situation in which there are only quantities and operations, all the relations are necessary. For example, if you put together two parts, the whole is the sum of the parts; if you take a part away from a whole, the result is the other part. These are examples of the logic of part-whole situations.

So far we have only given examples of logical relations in situations that involve addition and subtraction (i.e. additive relations), but there are also multiplicative logical relations, which involve the connection between multiplication and division. For example, in the problem “Mrs Brown had some stickers and she shared them equally among the 6 winners of a competition; each of the children received 3 stickers; how many stickers did Mrs Brown have?” the story is about an operation of division but the solution is obtained by a multiplication, which is the inverse of division. In this situation, there is a 3 to 1
correspondence between the stickers and the recipients, which defines the situation as involving the logic of multiplication and division: if we know the result of the division, which is 3 stickers for each of 6 children, we can find the quantity at the start by a multiplication. The inverse relation between multiplication and division is just as necessary as the inverse relation between addition and subtraction.

In contrast to necessary relations, contextual relations are the relations between quantities that are not connected by operations, but are set up in a description of the situation. They are not necessary because the description could have been set up in a different way. For example, given two quantities – Dylan has 15 marbles and Chris has 5 marbles – there is no necessary relation between the quantities, this is just how many marbles they have. We could create two different word problems about this situation by setting up two different descriptions of the relation between the quantities. One would be to say: “Dylan has 15 marbles; he has 3 times the number of marbles that Chris has; how many marbles does Chris have?” A second description would be: “Dylan has 15 marbles; he has 10 more marbles than Chris; how many marbles does Chris have?” Both problems make sense and there is no reason to think that one way of setting up the relation between the quantities makes more sense than the other. In school, we set up problems like these to help children think about relations between quantities that are not necessary; once the relation has been set up in the problem, the operations that have to be used to solve the problem can be identified by reasoning. Problems that involve relations offer an important opportunity for children to think about the connections between relations and operations, particularly because children find thinking about relations much more difficult that thinking about quantities, even when the problems can be solved by the same calculations.
There are many studies that have demonstrated that children find it more difficult to reason about relations than about quantities. We illustrate the difference between solving problems about quantities and about relations with the results of a study in the US, carried out by Hudson. The children were shown drawings, similar to the one in Figure 1, and asked to imagine that the birds were going to race to try to get the a worm. There were different pictures and the birds and worms were not lined in correspondence in any of the pictures. For some pictures, they were asked a question about a quantity: “How many birds won’t get worms? For other pictures, they were asked a question about a relation between the two quantities: “How many more birds than worms?”

![Figure 1](image)

a. How many birds won’t get worms?
b. How many more birds than worms?

The 7-year-olds’ answers to the question about the quantity (question a) were 100% correct; in contrast, only 64% of the answers to the question about the relation (question b) between the two quantities were correct. This is a very substantial difference that cannot be explained by the difficulty of the numerical calculation, which is the same for both the questions.
Problems that involve contextual relations are usually more difficult if the relations are known but the quantities are not, and the students have to calculate using relations. They are also more difficult if the number of quantities and relations is increased; for example, a problem with three relations and three quantities is more difficult than one with two relations and two quantities. Students must learn to think about such problems because they are not uncommon outside school and also because these are the sorts of problem that are used to prepare them for going beyond arithmetic and using algebra. To exemplify, we cite a problem that we used in our research: “Kimberley, Clement and William were in a team in a quiz show. Kimberley scored 39 points more than Clement. Clement scored 18 points more than William. The team scored a total of 261 points. How many points did each one score?”

In the problems presented so far, all the relations were set up in the problem description. Thus the students’ task is to analyse how these relations should be connected to operations that lead to the solution. But it is possible to present problems to students that require them to set up the relations themselves. Figure 2 shows an example of a problem in which the students themselves have to set up the relations between the numbers. This problem was given by Kath Hart and her colleagues to students in the UK; the figure shows the result for three age groups.
The results illustrate that many more students established an additive relation between the quantities than a multiplicative relation. They thought of the numbers 6 and 4, which are the measures in match sticks, as indicating that Mr Tall is 2 match sticks bigger than Mr Short, and used this relation to conclude that Mr Tall is then 2 paper clips taller than Mr Sort, i.e. 8 paper clips. However, they did not consider that the units – matchsticks and paperclips – are not of the same size. They needed to think that, given the information that Mr Short measures 4 match sticks and 6 paper clips, each matchstick corresponds to 1.5 paperclips. The difference of 2 match sticks, therefore, corresponds to a difference of 3 paperclips, which means that Mr Tall measures 9 paper clips (note that there are other routes to solution).

Some researchers argue that it is very important to present students with problems that require the students themselves to set up the relations between quantities because
outside school, when the problems are not given by the teachers, they will need to know how to set up the relations in order to solve the problems correctly. This is what is normally meant by modelling: knowing how to set up relations and then choose the operations to solve the problem. Whe the students set up the relations themselves, they should be aware of the need to check the results. After they find the answer, they should think of ways of checking it.

In summary, in order to understand children’s ability and difficulties with problem solving, we need to consider how they deal with the building blocks of quantitative reasoning: numbers, quantities, relations and operations. All of these are important and so is the children’s ability to coordinate them. Children find it more difficult to think about relations than about quantities. They also find it a challenge to connect relations between quantities with the operations required for solving problems. In the problem solving programme that we tested with the children in our research, we searched for ways to help the children think and talk about relations between quantities and how to make connections between relations and the operations. This programme is described in the subsequent sections and later we also present the results of the programme’s evaluation.

5. Helping students reason about relations between quantities

When setting up the programme to help the students improve their problem solving ability, we thought that they already have had a lot of opportunity to learn how to represent quantities with natural numbers. From the time they are in pre-school they are asked to say how many items are in groups of objects and later in school they learn to represent quantities with written numbers. Our aim in this programme was to promote the students’
understanding of relations between quantities and how to move from these relations to choosing the operations in solving problems. The first unit in the programme focuses on necessary relations between quantities and the second on contextual relations.

Research shows that students have difficulty with relations but there is good evidence that they do considerably better if they use visual, iconic representations instead of symbolic representations in order to make decisions about which operation to use in more difficult problems. For example, students were asked to calculate in one study the final score in a game in which players could win or lose points. The children were either asked to use cards of different colours to represent the points won and the points lost or they were asked to use paper and pencil, and use symbolic representation. The children who used cards had little difficulty in calculating the final score. If the player lost 5 points, then lost 3 points, then won 2 points, the child could use 5 red cards, then 3 red cards, then 2 yellow cards, to represent the points lost and won. After this, the children easily recognised that they could add the two lots of red cards, then take away the yellow cards, and they would know how many points were lost in the end. In contrast, if they were asked to represent the problem symbolically, children wrote either 5 or -5 for the first score, wrote -3 for the second game score, and then +2 for the final game; their final answer could be 4, obtained by calculating 5 –3 +2, or could be 2, if they considered the results of 5 – 3 to be zero, and then added the points won in the last game. The use of iconic representations helped the students to think about the relations between the two types of score, points won and points lost; they became aware that all points lost should be added together and that the points won and lost cancelled each other – i.e., they became explicitly
aware of the inverse relation between addition and subtraction (points won and points lost).

There is research also about how the use of iconic models facilitates reasoning about contextual relations. Children in years 5 and 6 are able to solve rather difficult problems, such as the one about Kimberly, Clement and William mentioned earlier on, using visual representations that help them think about the relations between the quantities. The research is so far scarce but we used these studies in the design of our teaching programme, which capitalises on the use of visual, iconic representations to help the children represent, think about and discuss relations between quantities in problem solving. They are then expected to reach a better understanding of how to deal with relations between quantities and become able to solve more difficult problems later without the support of the visual representations.

Our programme was designed for students in years 5 and 6. In the first part of the programme, we worked with the students on the understanding of the inverse relation between addition and subtraction both with natural and directed numbers (i.e. positive and negative). In the second part, we worked with the representation and discussion of contextual relations in more difficult problems. We taught the children two ways of representing relations, one using bars (inspired by the Singapore Model Method) and one using lines for showing correspondences in multiplicative reasoning problems (inspired by Streefland’s ratio table). In some problems, the relations were set out in the description of the problem, but in others the students had to set out the relations themselves using the information from the problem.
Leaning to use diagrams to represent relations is not a simple form of learning, and we are still investigating what are the best ways of teaching children to use diagrams. We have carried out four studies, three of which were brief interventions (carried out by Deborah Evans and three Masters students, Leo Chin Ying Petrina, Pinxiu Shen, and Akhila Pydah); the fourth study was our own investigation about how children use visual diagrams during problem solving, which together with part 1 was implemented over 15 lessons. These investigations helped us understand the steps that students have to make in order to learn to use the diagrams, the advantages that they gain from learning to use diagrams, and the outcomes for their problem solving skills.

6. The outcomes of the programme

All the analyses that we have made so far of the effectiveness of our programme take the same simple pattern. The analyses compared children who took part in the programme on problem solving – the experimental group – with those who did not – the control group. We also had a third group of children, who took part in a teaching programme about probability, and who were also included in these comparisons – the probability group. The reason for including the children who were learning about probability in the comparisons was that they were learning about many things that are important in problem solving. Like the children in the problem-solving group, they were learning to represent relations between quantities using diagrams, they were learning to read questions carefully to identify the relations in each problem, and they were learning to set out the contextual relations in probability problems before they decided what computations they would carry out. We wanted to know if, when these abilities are developed in the
context of probability problems, they could be used also in the solution of problems that do not involve probability. This is an important issue in mathematics education because we always want to know what are the chances of children using knowledge developed in one context in another, new context. We had no preconceived idea about whether this transfer of a more general ability in problem solving would be possible, from a specific context, probabilistic reasoning, to another context in which probability concepts do not play a part.

We expected that the three groups would have similar scores in the assessment that they took just before the problem solving programme began. We also expected that the problem-solving group would have better scores in problem solving than the control group after they had participated in the teaching sessions. As we have indicated, we did not know whether the problem-solving group would have better scores than the probability group after participating in the teaching; if the probability group learned at the same time something about probability concepts and something about problem solving, they might do almost as well as the problem-solving group after their programme, and better than the control group.

In the assessments that we gave to the children after the programme, there were two types of question. Some were about problems that involved games and other situations in which they had to think about quantities and relations. Other questions were about simple equations, which they could solve by using their understanding of necessary relations (e.g. \( c - 15 = -8; c = 7 \)) but which we had not included in the teaching sessions. There were only a few questions of each type so the scores that we analysed included the overall results in the different questions. The problems in the assessments were not identical to those included in the programme.
The expectations that we had about the comparisons between the problem solving and the control group turned out to be correct, and thus the programme was successful. An interesting result in these comparisons was that the spread of scores for the children in the problem-solving group was much smaller than the spread for the children in the control group; in the problem-solving group, the lowest scores were not as low as those in the control group. This suggests that the weakest children benefited from the programme, not just the strongest ones. The children who participated in the problem-solving programme started to out-perform the children in the control group after they had participated in five teaching sessions and this difference increased after they had ten teaching sessions. To our surprise, the gap between the groups remained the same between 10 and 15 sessions, and did not continue to increase with further teaching. This could well be the result of a concentrated effort in the schools to teach more mathematics to the children in the control group, whose performance improved then at the same rate as that of the problem solving group, because the last term of our programme was the term just before the children participated in the Key Stage tests.

We also compared the probability group with the control group in these problems, which were not at all related to probability concepts. We expected that this group would have improved their general problem solving skills, because they had also learned to think about relations and represent relations between quantities using diagrams. They had no teaching about negative numbers, though, and we were curious to know how they would perform in comparison to the other groups. One simple result is that they did almost as well as the problem solving group in most of the assessments after the teaching programme, and that the difference between the problem solving and the probability group was not
statistically significant in any of the comparisons. The mean scores for the problem-solving group were higher but there was a lot of overlap in the spread of the scores of these two groups. However, in most comparisons the probability group did better than the control group. After they had participated in 5 probability lessons, the difference between the probability and the control group was small and not statistically significant but after 10 sessions it was already significant. Similarly to the gap between the control group and the intervention group, this gap did not change after 10 sessions.

7. The organisation of the description in the subsequent sections

The remaining part of this handbook presents somewhat detailed descriptions of the activities and some samples of the children’s work, with comments about the challenges and successes experienced by the children. The first section presents general principles used in the design and carrying out of the activities. The children should be active the whole time; the teacher’s role in these activities is to explain the activities, raise questions, make sure that all the children are engaged in solving problems and have opportunities to explain their answers at some point, and summarise with the children’s participation what they have accomplished. The teacher plays a major role in helping the children to become aware of the connections between the activities, summarise what they have learned, and identify different ways of solving the same problem. This intellectual leadership from the teacher will be best accomplished if the teacher has given much thought to the activities and the children’s possible reactions to them. We learned a lot during the running of the programme each time we ran it and we are confident that the teachers using these materials will too. Our attempt to anticipate what might happen in your class based on our
experience will have to be complemented by your observations of what does happen in your own class. We learned that, at the end of each session, it was a good idea to look at what each child had produced and what seemed to need some revision at the start of the next session.

8. Final words

We hope that this description of the building blocks for thinking mathematically and of research that shows why we developed our programme in this particular way shows why this new approach to teaching children about problem solving is useful. In our research we taught children first about necessary relations because a substantial minority of children finds it difficult to use the inverse relation between addition and subtraction during problem solving. Most children in Year 5 struggle with this inverse relation in the context of negative numbers, even if they are at ease with it when thinking about natural numbers. We learned through our research that year 5 and 6 students can learn to represent relations explicitly using diagrams but that this is not as easy as an adult might imagine. A diagram is not produced by following rules blindly; the children must reason about the relations between the quantities in order to produce a helpful diagram. Sometimes we wandered whether the children drew the diagrams correctly because they had already solved the problem, but their attempts to find the right diagram and their discussions when they presented their work showed that while doing the diagram they thought more about the relations between the quantities.

The results were very encouraging, and it seems to us that the children really did learn some worthwhile ways of approaching problems, of thinking about the problems before deciding what calculation to carry out, and not jumping to calculations without
reflecting. In the rest of this handbook we shall be describing how we set about this teaching, and how the methods that we developed with small groups of children could be adapted to classroom teaching. We want to find out now whether our teaching programme is also effective when it is carried out in school classrooms. That is our next step.

We very much hope that you contribute to the next step in searching for new and effective ways of teaching students about problem solving.
General principles we used in the interventions

- The children should always be actively solving problems. Each of them should produce an answer for every problem. The plenary discussion takes place after each one has answered the question.

- The children’s reasoning is supported throughout the programme in different ways, but our emphasis is on the use of visual representations that help them think about the relations between quantities. This is sometimes implemented by the use of cards of different colours to indicate points won or lost, by arrows that show they are trying to find out what was the situation before something happened, or by the use of diagrams. In many of the problems, the aim is to arrive at the calculation that would lead to solving the problem, and we asked the children what calculation they should carry out rather than the answer. When the numbers are larger and the reasoning demanding, we used calculators so that the children’s attention could be focused on the reasoning.

- Discussing their answers is a key element in this project. It is therefore useful for the pairs of children to write an explanation of how they reasoned about the problem. Writing explanations in pairs helps children who might be able to explain their reasoning orally but find writing a challenge. Children should be invited to explain their reasoning even if they did not arrive at the correct answer. In some of the more difficult problems, they may have reasoned well about the relations between quantities but missed a part of the problem. They can become more confident in their reasoning and learn to check the solution process better before deciding that they have solved the problem.

- Even when all the children made mistakes, we avoid simply telling them the answer, as this will be of little help for their reasoning. We can start a solution with the visual representations and then see whether they can continue it, or we can pose a question that they could have asked themselves: for example, did you have more or fewer points than before you lost 3 in the game? If you now have fewer, how do you get back to the number you had before?

- Many activities were designed to be carried out with children in pairs to give them the chance of making their reasoning explicit in the discussion with the other child. Only then would they be in the position to explain their reasoning to the whole group. But one must watch out and not allow one child in the pair to always solve the problems first all the time and just explain the solution to the other one. This is why in many activities the children are asked to take turns and the teacher will need to watch the pairs as they work.
The programme for teaching problem solving

Overview

There are two units, which together take 14 lessons.

1. **Unit 1** aims to help children develop their understanding of necessary relations in problems, with special emphasis on the inverse relation between addition and subtraction. The children are given plenty of opportunities to use numbers that represent quantities, and think about relations between quantities, but also to think about necessary relations between numbers. This unit takes about half of the teaching sessions.

2. **Unit 2** aims to develop children’s ability to think about contextual relations. In some problems, the contextual relations are set out in the description of the situation and in others the children must set out the relations themselves. In order to help them think about and discuss contextual relations, the children are introduced to the use of bar and line diagrams.
Unit 1

Session 1

Marbles game: understanding the relations between positive and negative numbers
Aim: to help children think about how to combine positive and negative numbers. In the process of putting together the points won and points lost, the children phrase the relation between these two types of values in many ways, and explicitly mention general rules such as you need to add all the points won and lost separately and the points won and lost cancel each other (the inverse relation).

This first activity presents a series of direct problems in a context of games where children can win or lose marbles. They are direct problems because the end result is the unknown.

What do you need?
- 15 red and 15 yellow cards for each pair (not included in the resources pack)
- children’s booklet to record their reasoning

What do you do?
- Present the children the yellow and red cards that they have in front of each pair. These are used to represent the points won and lost, helping them to solve the problem.
- Read the first problem, asking the children to record the numbers presented:

**Problem 1:** “Theo played four marbles game. In the first game he won 1 marble, in the next he won 4 points, in the next he lost 2 and then he won 6 marbles. In the end, had he won or lost marbles? How many?” (Answer: won 9)

- Show the group how to use the cards to represent the marbles:
  - Decide with the children the colour of the cards for the marbles won and the lost (e.g. the yellow cards can be the marbles won and the red cards the marbles lost)
  - Ask the children to put down the number of cards corresponding to the marbles won and the marbles lost (i.e. yellow cards: 1, 4 and 6; red cards: 2).
  - Ask the children to find out whether Theo had won or lost marbles at the end, and how many.
  - Ask the children to write this down for number 1. The children must always indicate whether it was a positive or negative outcome. They can use the word (won 9) or numerical signs (+9) depending of whether you have taught this earlier or not. You can introduce the convention at the end of the first set of games, if you wish.
- Discuss with the children their solutions, asking them to explain how they solved the problems. Some children will put all the cards of the same colour together, carry out a calculation (e.g. lost 2, then take away 2 yellow cards and remove the 2 yellow cards) and then say the final answer; some children will carry out the computations as they go. Some will have cards of the same colour “cancel each other” and count the remaining cards. Some count backwards, a procedure that sometimes leads to mistakes as they may skip zero (go from 1 to minus 1). It is worthwhile for them to see that there are different ways of getting the answer, so that those who are using awkward procedures watch other using simpler procedures, which they may understand and then adopt.
- Read each of the subsequent problems aloud. The children should find the solution by themselves before beginning the discussion with the whole group.
Problem 2: There was another marbles game and Laura won 1 marble, then won 3, then lost 6 and then lost 4. At the end, had she won or lost marbles? How many? (Answer: -6)

Problem 3: Chloe was playing a game and lost 2 marbles, won 6, lost 6 and won 3. At the end, had she won or lost marbles? How many? (Answer: +1)

Problem 4: Adam was playing a game and lost 1 marble, won 3, lost 6 and lost 2. At the end, had he won or lost marbles? How many? (Answer: -6)

Problem 5: Lucy was playing a game and won 1 marble, lost 4, lost 2 and won 3. At the end, had she won or lost marbles? How many? (Answer: -2)

Problem 6: John was playing a game and lost 6 marbles, won 3, lost 3 and won 1. At the end, had he won or lost marbles? How many? (Answer: -5)

Problem 7: Chris was playing a game and won 7, then he won other 2 and then he lost 3 and won 5. At the end, had he won or lost marbles? How many? (Answer: +11)

Problem 8: Sam was playing a game and won 2, then won another 9, then lost 3 and lost 1. At the end, had he won or lost marbles? How many? (Answer: +7)

Problem 9: There was a game and Jack lost 4 marbles, won 7, lost 5 and won 8. At the end, had he won or lost marbles? How many? (Answer: +6)

Problem 10: Hannah was playing a game. She lost 3 marbles, won 8, lost 2 and lost 6. At the end, had she won or lost marbles? How many? (Answer: -3)

Problem 11: Helen was playing a game. She won 1 marble, lost 9, then lost 4 and then won 3. At the end, had she won or lost marbles? How many? (Answer: -9)

Problem 12: Sarah was playing a game. She lost 8 marbles, then she won 3, then she lost 7 and won 2. At the end, had she won or lost marbles? How many? (Answer: -10)

Watch out!
It is possible that children will have more difficulties when the first number given is negative in comparison to when the child playing wins marbles in the first game. Some children convert the initial loss into zero, as the child did not win anything. The use of cards is particularly helpful here.

They can use the cards in different ways – for example:
- add all the cards won, add the cards lost and then subtract one set from the other;
- cancel partial results as they progress (won 3, lost 2, they take 2 away instead of adding 2 cards of the other colour);
- count backwards as they proceed.

It is a good opportunity for comparing procedures and validating different ways of arriving at the answer.

In some groups of children, this may go very quickly and you may be able to move forward to the next activity in the same session.
Session 2

Computer game level 1 and 2: using the inverse relation between addition and subtraction to solve problems

Aim: help children to understand or be explicit about the concept of inversion between addition and subtraction.

There are 20 problems divided in two levels to make this activity more similar to a computer game. Some of them are direct problems and others are inverse problems. The children are asked to write down the calculation they would enter in a calculator in order to come up with the solution.

All the problems use small numbers because often the children use an addition that they know to solve a subtraction problem (e.g. in an inverse problem in which they should enter 7-2 to find the answer 5, they enter the operation 5+2; they need to become aware of the difference between knowing the answer and entering the right operation in the calculator).

What do you need?
- Computer to show the presentation of the 20 problems (file titled: 'Computer game level 1 and 2', found in the computer Materials folder for Unit 1, Session 2)
- Children's booklets to record their solutions
- A printed list of the questions for the children to move on in pairs once the task has been discussed and understood (see Appendix A or file titled: 'Computer Game Level 1 and 2_questions', found in printable resources folder in Unit 1, Session 2)

What do you do?
Read each of the 20 problems to the whole group while showing on the Computer the slide related to the problem.

The children need to:
  - write in their booklet the calculation that they should enter in a calculator to find the answer.

After the children say their answers, in the inverse problems, ask the children to explain why they used addition or subtraction (for example, in a problem where the quantity decreased, why did they use an addition?)

  - If in the inverse problems the children write the wrong calculation, help them to:
    - reason whether, before the change in quantity, there should be more or less objects than they will have at the end;
    - imagine going backward moving from the end of the problem towards the beginning.
  - Once the calculation is agreed, show the feedback on the computer screen.

Use the computer screen for feedback and for discussing the commutative property of addition and the non-commutativity of subtraction.

In the problems that use addition, discuss with the children the commutative property of addition: “Is there any difference between entering 5+4 and 4+5?” Discussion the non-commutativity of subtraction: “Is there a difference between 7-5 and 5-7?”
Watch out!
There will be large individual differences and some children may still be struggling with the inverse problems whereas others find them quite easy. Once you have discussed the first 5 or 6 items and have made sure that the children understand that you are not asking what the answer is but what they would enter in a calculator, the pairs can move on independently using their own screen and you can work with those who find it more difficult. At the end, you can check all the answers as a group. The pairs exchange booklets so they check whether other children have written the right calculation and count the points that the other pair achieved.

Dice game

Aim: to give the children more practice in thinking about quantities marked as positive or negative.

What do you need?
- Dice with numbers: one for each pair
- Dice with only plus and minus drawn on them
- Children’s booklet to record their answers

What do you do?
- Explain that the symbol ‘plus’ on the die means that you win points whereas the symbol ‘minus’ means you lose points.
- Explain that each player in the pair throws the two dice four times to obtain four numbers and his/her peer writes them in his/her booklet; you can agree with the children on the convention of a mixture of words and numbers (e.g. won 4) or the use of mathematical signs only (+4), depending on how good the children are at using the symbolic representation. Each set of four throws is one game.

The children should:
- Keep record of the four outcomes obtained by writing them in the booklet.
- Find the outcome for each turn of four throws for each child and compare the outcomes, marking the winner in the pair for each game.
- The pairs can have five games and then find out who won over the five games.
- Write the calculation of the outcome of the five games on a transparency and be ready to discuss with the whole group how they found out the final score, over the series of games.

If the children have used different forms of representation, this is a good opportunity for comparing their ways of representing the points won and lost.
Session 3

**Computer game level 3 and 4: using the inverse relation between addition and subtraction to solve problems**

Aim: this activity is similar to the one presented in Session 2 but now the children cannot use their knowledge of results when entering the operation in the calculator because the numbers are most likely to be outside the known number bonds. As in the previous activity, the children must reason about the problem in order to decide which operation is the correct one to find the solution. Calculators are provided so that the children can write the calculation and also the answer.

**What do you need?**
- Computer for teacher-led presentation of the 20 big numbers problems (titled: ‘Computer game level 3 and 4’, found in Computer Materials for Unit 1, Session 3)
- Calculators: one for each pair of children
- Children’s booklets to record their answers
- A printed list of the questions for each pair so they can move on the task when the task is understood (see Appendix A or file titled: ‘Computer game level 3 and 4 Questions’, found in the printable resources folder, Unit 1, Session 3)

**What do you do?**
Apply the same procedure presented in Session 2. As in the previous session, after you have made sure that the children know that they should be writing what computation will lead to the right answer, they can then proceed independently in pairs. After they have finished, they can exchange booklets with another pair who will check their answers.

Remember to discuss their reasoning with them and elicit explicit explanations of the inverse relation between addition and subtraction. You can also revisit the commutativity of addition and non-commutativity of subtraction.

**Watch out!**
We suggest that the children should not be allowed to erase their answers so you can go through the booklets in the end. Some children succeed in all direct problems and seem to be at chance level in the inverse problems. It would be important to do some supplementary work with these children.

**Gremlins game level 1: inversion problems with negative numbers**

Aim: to promote thinking about the inverse relation between addition and subtraction in the context of directed numbers. It is surprising that many children who seemed very competent in the previous activity will find this one a challenge.

There are 12 problems with an unknown beginning, a known transformation and a result. The children need to find the starting point.

The questions have different levels of difficulty, which are mixed across the games labelled Level 1, Level 2 etc. A final game with three transformations that have to be inverted always causes great difficulty even after the children are confident with the inversion of one transformation.
What do you need?

- Computer (one for each pair) to present the ‘Gremlins game level 1’ (file titled: ‘Gremlins game level 1 presentation’)
- Teacher’s computer to show how the game works and present the feedback for each game (file titled: “Feedback for Gremlins game level 1”)
- Children’s booklet to record their answers

What do you do?

- Explain to the whole group the aim of the game and how to record their answers following the presentation of the gremlin game level 1.
- Go through the practice example with the whole group discussing how they can reach the solution (remind them of the marbles and dice games):
  - Ask them if they should have more or fewer points at the beginning of the game than at the end;
  - Ask them to imagine how to go backward and put the points back (before a transformation in which points were lost) or take points away (before a transformation in which points were won).
- Once the practice example is discussed, each pair can play independently.
- After each problem is solved, ask individual children to explain how their reached that solution and ask whether there were other ways to solve the problem.
- Show the feedback on your computer.

Watch out!

- If some children are faster than others, in order to keep them working they can earn extra points by writing good explanations about how to find the answer.
- It is likely that children will have more difficulty when a negative score in the second game leads to a positive final results or a positive score in the second game leads to a final negative score. Some children can explain this very well in their own words, so giving them the opportunity to explain how they thought is very helpful. For example, one child said: “If he lost 5 points and is still winning 3 points, that means he had a bigger winning score before, and it had to be bigger by the 5 points that he lost.” Another child said: “You have to put back the points that he lost, so if he has 3 and lost 5, and you put the 5 back, then you know he had 8 before.”
- If all the children are struggling at the beginning, you can ask questions that help them connect this game with their previous activities with the computer:
  - At the beginning of the game, did the player have more or fewer than at the end of the game?
  - If you were to put back the points he lost (or take away the points he won), what would be the score at the beginning?
Session 4

**Gremlins game level 2: inversion problems with negative numbers**

Aim: To give the children more opportunity to think about the inverse relation between addition and subtraction in the context of directed numbers. The level of challenge is increased by having items that are more demanding (as described earlier on).

This activity is continues from the previous ones and is run in the same way as in Session 3.

**What do you need?**
- Computer (one for each pair) to present the game (file titled: ‘Gremlins game level 2 presentation’, found in Computer Materials for Unit 1, Session 4)
- Teacher’s computer to present the feedback for each game (file titles ‘Feedback for gremlins game level 2’, found in Computer Materials for Unit 1, Session 4)
- Children’s booklet to record their answers

**What do you do?**
See instructions presented in Session 3.

**Gremlins dice game**

Aim: To introduce the comparison of two directed numbers.

In this activity, the meaning of the Gremlins and Spaceship hits are the same. It is a useful activity at the end of the session as something slightly easier than the computer games where the starting point is missing.

**What do you need?**
- Dice with numbers: one for each pair
- Dice with gremlins and spaceships: one for each pair
- Children’s booklet to record answers

**What do you do?**
- Explain that the symbol of the space ship ⚫ means that you lose points whereas the symbol of gremlin ⚫ means that you win points.
- Each player in the pair needs to throw both dice (one with gremlins and spaceships and one with numbers) four times to obtain four numbers (for example “gremlin 4” means you win 4 points: “spaceship 2” means you lose 2 points).
- Keep record of the four numbers obtained writing them in the booklet.
- Each player has to determine how many points they have at the end of the first four throws.
- The player who has more points after these four throws wins the game.
- The game continues like this for other five times.
- Ask some children to show how they calculated the final score and discuss whether there are other ways of calculating.
- Ask the children whether someone can win the game with a score of zero.
Session 5

Gremlins game level 3: inversion problems with negative numbers

Aim: To increase the level of difficulty in the inverse problems and give the children more practice with the inverse relation between addition and subtraction in the context of directed numbers.

What do you need?
- Computer (one for each pair) to present the game (file titled: ‘Gremlins game level 3 presentation’, found in Computer Materials for Unit 1, Session 5)
- Teacher’s computer to present the feedback for each game (file titled: ‘Feedback for gremlins game level 3’, found in Computer Materials for Unit 1, Session 5)
- Children’s booklet to record their answer

What do you do?
See instructions presented in Session 3.

Number Scrabble

Aim: To provide an opportunity for exploring different ways of composing the same number. Everyone finds it very easy to think of 5, for example, as 4+1 or 3+2 but it can also be thought of as 8-3 or 7-2.

What do you need?
- Cards with positive and negative numbers
- Cards with numbers and arrows (up or down)

What do you do?
- Show the two different types of cards:
  - The cards with positive and negative numbers mean that you are winning or losing points respectively.
  - The cards with the arrows and numbers mean that you are counting up (i.e. the values increase) or down (i.e. the values decrease).
- Each player takes 5 of the number cards and 5 of the arrow cards.
- The first player lays down a correct sequence of 3 cards: a number card, an arrow card and a number card (this will be the total of the other two cards). For example:
  
  
  
  
  This means that you had lost 8 points, then you counted up 3 and, as result, you now are losing 5 points. If the first player can make a sentence, he/she wins a point. If the first player says he/she doesn’t have cards to make a correct sentence, and the second player sees a solution, the second player can make the sentence and win a point.
- The other player tries to use one of the cards already laid down combined with two of their own cards, to create a new sequence. The sequence has always to have one number card.
one arrow card and another card. The second player can use any of the cards already on the table. For example:

```
+6
-8 ↑3 -5
+3
```

- If the second player can make a correct sentence, he/she wins a point. If the second player can’t see a solution and the first player can, the first player can make the sentence and win a point.
- The children take turns being the first one to make a sentence.
- So, for each go, one plays has the first attempt but the other one can win a point by finding a solution that the player did not see. The maximum score for each round is 2 points. The round is then finished and a new set of 5 cards with numbers and five cards with arrows is drawn out.
- The winner is the player with the biggest number of points.
- The children should keep a record of the sentences with the players’ name on the sentences they created.

**Watch out!**

- It is more difficult to explain than to demonstrate this game. The teacher can explain the game by drawing the 5 number and 5 arrow cards, writing them on the board, and asking the children to try to make a sentence. One this is done, the teacher draws the other 5 number and 5 arrow cards, writes them on the board, and again asks the children to try to make a sentence using one of the cards already on the board.
- If the children find difficult to count up or down, a number line can be made available.
Session 6

Detective game level 1

Aim: to provide opportunities for practice with the inverse relation between operations without reference to quantities.

What you need for each pair of children

- Envelopes into which to put the cards with the questions
- Print cards with the questions for the detective game (file titled: ‘Cards for the detective game Level 1, found in printable resources for Unit 1, Session 6)
- Print badges with constable symbol (file titled: ‘Detective game symbols to print, found in printable resources for Unit 1, Session 6)
- Children’s booklets to record their answers

What do you do?

- The story is: “A criminal hid a potent weapon in a safe and the children are helping the police to find out the numbers for the combination of the safe”.
- Give each child the badge with their names and to each pair an envelope with the 12 questions cards.
- Explain that each pair has to find the number that the criminal was thinking of using the clues in each question.
- Each pair can work together on the first 2-3 questions. After this, the whole group can discuss their answers explaining their reasoning.
- If some pairs finish very quickly:
  - Children can insert the numbers found in the safe and move the gear backward or forward. Positive numbers mean that they have to move the gear forward, negative numbers means that the gear moves backward.
  - After the first movement, the second movement has to refer to the previous one. For example: if in the first movement the gear stops at -15, and the second number is 3, the gear will move to -12.

Questions

1. The criminal thought of a number. Then he added 6. The result was 1. What number was he thinking of? (Answer: -5)

2. The criminal thought of a number. Then he took away 4. The result was 3. What number was he thinking of? (Answer: 7)

3. The criminal thought of a number. Then he added 1. The result was -3. What number was he thinking of? (Answer: -4)

4. The criminal thought of a number. Then he took away 4. The result was -7. What number was he thinking of? (Answer: -3)

5. The criminal thought of a number. Then he took away 2. The result was 8. What number was he thinking of? (Answer: 10)
6. The criminal thought of a number. Then he added 9. The result was -7. What number was he thinking of? (Answer: -16)

7. The criminal thought about a number. Then he added 2. The result was 5. What number was he thinking of? (Answer: 3)

8. The criminal thought of a number. Then he took away 7. The result was -9. What number was he thinking of? (Answer: -2)

9. The criminal thought of a number. Then he took away 8. The result was 4. What number was he thinking of? (Answer: 12)

10. The criminal thought of a number. Then he added 5. The result was 8. What number was he thinking of? (Answer: 3)

11. The criminal thought of a number. Then he added 10. The result was 2. What number was he thinking of? (Answer: -8)

12. The criminal thought of a number. Then he took away 6. The result was -11. What number was he thinking of? (Answer: -5)

Math puzzle 1

Aim: To provide more practice on the inverse relations between operations.

What do you need?
- Print page titled “Where is the second safe?” (file titled: ‘Second safe_detective game’ found in printable resources for Unit 1, Session 6)

What do you do?
- When the children have finished solving all the questions and moved the gear, tell them: “In the first safe the police found this page. This contains a code that will tell you where to find the second safe”.
- Children need to solve the number sentences: some are direct and some are inverse so that the children keep thinking of when to use the inverse relation between addition and subtraction.
- The numbers found for each equation corresponds to a letter.
- If they solve all the equations, they will decode the sentence: “under the police station”.

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Gremlins dice game

Aim: To give further opportunities to think about combining and comparing directed numbers. In the previous gremlins dice games, the children only needed to be able to know which number was bigger than the other. They now need to know the difference between the numbers. Finding the score in the game and finding the difference between scores require different reasoning about the numbers. Note that we are not using calculation terms (adding or subtracting directed numbers) but referring the students to the situation so they realise what they need to do with the numbers.

What do you need?
- Dice with numbers
- Dice with drawn gremlins and spaceships
- Children’s booklet where to record their answers

What do you do?
- Explain that the symbol of the space ship means that you lost points whereas the symbol of gremlin means that you win points.

Note a change of rule: if the difference in the outcome of each throw is 3 or less, the turn is considered a tie. One player only wins if the difference between the scores is more than 3 points.

What the children do
- Each player in the pair needs to throw both dice four times, to obtain four numbers plus a symbol, gremlin or spaceship.
- The other player keeps record of the four numbers obtained by writing the outcomes in the booklet and calculates the result for the turn.
- The players have to determine how many points they have at the end of the four throws.
- The pair now compares their final scores: if a player has more than 3 points than the other player, he/she wins one point.
- The game continues like this for five turns per player.
- Discuss with the group whether you can win with a score lower than zero.
Session 7

Run away game
Aim: discuss with the children the relation between numbers in the context of positive and negative numbers.

This activity is in direct contrast with the reasoning used in the marbles and scoring in the gremlins games. It is similar to the comparison of scores in the gremlins game above. In order to solve the problems, the children need to add the numbers when they have opposite signs and subtract them when they have the same sign. This variation in how the numbers are treated because of the situation and the question they are asked is an important aspect of reasoning to solve problems.

What do you need for each pair?
- Two characters for each child
- One die with numbers
- One die with plus and minus
- Board with scenario for the game with number line and without number line
- Children’s booklets to record their answers

What do you do?
- The story: “Two children were in the science laboratory doing some experiments when an explosion happened and created a sticky material. In order not to be stuck together, they had to run away as far as possible.”

What the children do
Each player in a pair needs to:
- Choose two characters each and put them in the starting point (in the laboratory)
- Throw the two dice (number die and the plus/minus die) twice to obtain two directed numbers: one for the first character and one for the second character.
- Record their throws in their booklet
- Work out the distance between the two characters

- The player whose characters are more than 4 points apart from each other receives one point.
- If the characters are less than or 4 points apart from each other, they don’t score any points because they stick together.

Play the first game using the scenario with the number line under the laboratory and then give the children the scenario without the number line.

Discuss with the children some of their games focusing on clarifying the reasons why it is necessary to add or subtract the numbers.
Watch out!
We noticed that many children attempted to develop and use rules instead of paying attention to the questions they were trying to answer. This was the motive to create situations in which they had to think about the numbers in different ways.

Detective game level 2
Aim: To help the children to solve inverse problems with positive and negative numbers with two transformations. These problems present an unknown beginning, two transformations and a result. Children find them more difficult than those with one transformation because they require thinking about intermediary results. You can remind them to imagine moving backward from the result to the beginning and that they can write the intermediary result on their booklets.

What you need for each pair?
- Envelopes for the cards with the questions
- Print cards with questions for the game (file titled: ‘Cards for detective game _Level 2, found in printable resources for Unit 1, Session 7)
- Badges with inspector symbol, (file titled :Detective games symbol to print, found in printable resources for Unit 1, Session 7)
- Children’s booklets to record their answers

What do you do?
- Give children the new badges of inspectors and the new 12 questions.
- Children have to find the numbers for the combination of the safe using the clues presented in the cards.
- If some children are quicker than others, they can transform the numbers to move the gear of the safe as they did for the activity in Session 6.

Questions
1. The criminal thought of a number. Then he added 6 and took away 2. The result was 1. What number was he thinking of? (Answer: -3)
2. The criminal thought of a number. Then he took away 4 and then another 3. The result was -3. What number was he thinking of? (Answer: 4)
3. The criminal thought of a number. Then he added 1 and then another 5. The result was 7. What number was he thinking of? (Answer: 1)
4. The criminal thought of a number. Then he took away 7 and then added 2. The result was -8. What number was he thinking of? (Answer: -3)
5. The criminal thought of a number. Then he took away 2 and then another 8. The result was 3. What number was he thinking of? (Answer: 13)
6. The criminal thought of a number. Then he added 9 and then another 5. The result was 4. What number was he thinking of? (Answer: -10)

7. The criminal thought of a number. Then he took away 6 and then added 3. The result was -2. What number was he thinking of? (Answer: 1)

8. The criminal thought of a number. Then he added 1 and then took away 7. The result was -11. What number was he thinking of? (Answer: -5)

9. The criminal thought of a number. Then he took away 8 and then added 5. The result was 9. What number was he thinking of? (Answer: 12)

10. The criminal thought of a number. Then he added 5 and then another 7. The result was 4. What number was he thinking of? (Answer: -8)

11. The criminal thought of a number. Then he took away 3 and then another 7. The result was -12. What number was he thinking of? (Answer: -2)

12. The criminal thought of a number. Then he added 5 and then took away 4. The result was 8. What number was he thinking of? (Answer: 7)

Watch out!
Some children tend to solve first the two transformations and then compare the number they obtained with the result to find the number at the beginning. This procedure is error prone. For example, in the last problem, they would do 5 take away 4 is 1, but the answer is 8; they are unsure whether they should add 1 to 8 or subtract 1 from 8. If they start guessing, they can get some answers right and others wrong but they won’t have reasoned about the problem all the way. Some children don’t mind getting only some of the answers right and others wrong. Remind them to think that they can invert each operation and write down the intermediary results. They could also invert the result (i.e., 5 take away 4 is 1, the inverse is -1, so they subtract 1 from 8) but not all children understand this procedure and treat it as a rule.
**Math puzzle 2**
Aim: To provide more opportunities for the children to work with the inverse relations between operations using more transformations.

**What do you need?**
- Print page titled “Which is the password for the third safe?” (file titled: ‘Third safe_detective game’, found in printable resources for Unit 1, Session 7)

**What do you do?**
- The story: “In the second safe, the police found another message that the children have to decode. If they decode all the number sentences, they will have the password for the third safe”

**What the children do**
- Children need to solve the number sentences.
- Each number corresponds to a letter.
- The sentence that they have to decode is: “think about inversion”.
Session 8

Detective game level 3
Aim: To help the children to solve inverse problems with three transformations.
This activity is the most difficult up to now. Therefore, it is even more important to stress that, in order to solve the problems, it is necessary to imagine going backward from the result to the beginning of the problem.

What you need for each pair?
- Envelopes to put the cards with the questions
- Print cards with the questions (file titled: ‘Cards for detective fame _level 3 , found in printable resources for Unit 1, Session 8)
- Print badges with chief inspector symbol (file titled; ‘Detective game symbols to print’, found in printable resources for Unit 1, Session 8)
- Children’s booklets to record their answers

What do you do?
- Give to each child the new badges of chief inspector and to each pair a new set of 12 cards.
- Tell the children that this is the last safe and they have to find the numbers for the combination of the third safe using the clues provided in the questions.
- If some children are quicker than others in solving the problems, they can use the numbers to move the gear of the safe (negative numbers move counterclockwise, positive numbers move clockwise).

Questions
1. The criminal thought of a number. Then he added 3, then another 4 and then another 2. The result was 11. What number was he thinking of? (Answer: 2)
2. The criminal thought of a number. Then he took away 5 and then another 1 and then another 6. The result was -16. What number was he thinking of? (Answer: -4)
3. The criminal thought of a number. Then he added 9 and then took away 2 and took away another 3. The result was -3. What number was he thinking of? (Answer: -7)
4. The criminal thought of a number. Then he took away 10 and then another 1, and then he added 7. The result was 5. What number was he thinking of? (Answer: 9)
5. The criminal thought of a number. Then he added 8 and then other 3 and then he took away 3. The result was -4. What number was he thinking of? (Answer: -12)
6. The criminal thought of a number. Then he took away 6, then another 5 and then another 4. The result was 2. What number was he thinking of? (Answer: 17)
7. The criminal thought of a number. Then he added 2, then another 6 and then added 1 more. The result was -7. What number was he thinking of? (Answer: -16)

8. The criminal thought of a number. Then he took away 4 and then added 7. Then he took away 5. The result was 6. What number was he thinking of? (Answer: 8)

9. The criminal thought of a number. Then he added 7 and then took away 8. Then he added 8. The result was -4. What number was he thinking of? (Answer: -11)

10. The criminal thought of a number. Then he took away 1 and then added 9 and then added 3. The result was 8. What number was he thinking of? (Answer: -3)

11. The criminal thought of a number. Then he added 5 and then added 3 and then added 2. The result was 10. What number was he thinking of? (Answer: 0)

12. The criminal thought of a number. Then he took away 10 then added 2 and then took away 9. The result was -9. What number was he thinking of? (Answer: 8)

Run away game 2
Aim: to discuss with the children different relations between numbers in the context of positive and negative numbers.

What do you need for each pair?
- Two characters for each child
- One die with numbers
- One die with plus and minus
- Board with scenario for the game with number line and without number line
- Children’s booklets to record their answers

What do you do?
Follow the instructions for the run away game presented in session 7.
Unit 2

The aim of the sessions in Unit 2 is to introduce to the children the use of diagrams in problem solving to represent contextual relations. The children will be encouraged to draw a diagram first and to think about a computation only after they consider the relations between the quantities in the diagram. We have chosen two diagrams for this programme, one that uses bars to represent quantities and one that uses parallel lines to represent quantities that are in correspondence with each other.

By the end of this unit students should be able to draw bar diagrams for additive reasoning and ratio diagrams for multiplicative reasoning problems and use them effectively. Sometimes they use lines instead of bars for additive reasoning problems but do not seem to confuse the forms of reasoning. When they do offer an additive solution for a multiplicative problem (or vice versa, but this is less common), they should be asked questions that help them see why their answer did not work.

- The emphasis is on the children’s reasoning about the problems and not on their calculation abilities.
- Drawing a diagram to solve a problem enables students to describe the situation in the problem and visualize the relations. This should help them to keep track of the steps they have taken to solution when there are more steps.
- The students are asked to explain how they reached their answers to each problem by presenting their diagrams and solutions on transparencies using the overhead projector. The whole group is encouraged to give feedback to the students presenting. The group is asked to consider if it is possible to solve each problem in a different way.
- Some examples of diagrams and students’ productions are included as illustrations.

For all 5 session you will need:

- Overhead projector
- Transparencies
- Calculators

Some problems are presented on the computer screen and others are presented on cards, which will allow the pairs to move at different rates. In the first sessions, we used computer presentations so the pairs kept moving at comparable pace and the answers were discussed in whole class sessions. Later on, the pairs can move at their own pace but at some points you will call the students into a whole class session to discuss the use of diagrams and compare solutions.
Session 1

How can we represent a problem?
Aim: To enable the children to attempt to use visual, iconic representations of contextual relations between quantities. We opted for two types of diagram that we expect will support different discussions about the relations between the quantities. It is crucial that the children should attempt to use a diagram, explain what their diagram shows and how it helps them solve the problem. They should also think of ways of checking whether their solution makes sense within the context of the information they have.

We introduced the diagrams but did not guide the children in a step-by-step manner in their use. We allowed the students to explore their diagrams and compare them with those produced by other students. It is possible to work with diagrams differently, and guide the children in a step-by-step manner, at least in the beginning of this process. The discussion of the different diagrams and solutions is important for the students to become aware of the relations they are attempting to represent.

Additive reasoning problems will be represented using bars for the quantities

What do you need?
- A computer to present the problems (file titled: ‘Presentation for word problems_ Unit2_Session1’, found in computer material for Unite 2)
- Children’s booklets to record their diagrams and answers

What do you do?
- Present the first problem on the screen and read it to the whole group
  “Theo and Danny are exchanging stamps. They have the same amount of stamps in their collection. Theo gives 5 stamps to Danny. Does Theo have more or fewer stamps than before? How many more stamps does Danny have than Theo?”
  
- There should be two answers coming from the children, 5 and 10. After the children have answered the questions, you can present the diagram with the bars on the computer step by step:
  - First students are asked to think who the characters in this problem are and how many stamps they have at the beginning. Two strips of the same height are drawn, one for each of the character, representing the number of stamps they have at the beginning (presented on the computer screen). Emphasise that the bars are of the same height because they have the same amount of stamps.
Ask the children what is the next piece of information given in the problem (we know that Theo gave 5 stamps to Danny). A piece of the bar representing the 5 stamps is cut from Theo’s bar and is added onto Danny’s bar (presented on the computer screen):

Ask them again: “So how many more stamps does Danny have than Theo?”

For the next problems, the children should draw the bars to represent the quantities, and show how they solve the problems. Each child should solve each problem, then check with the peer with whom they are working, and then agree on a diagram and answer that will be written on an acetate for presentation to the class.

**Problem 1:** Clare and George have the same amount of books in their library and now they will be exchanging books. George gave 10 books to Clare. Does George have more or fewer books than before? How many more books does Clare have than George?

**Problem 2:** Eric, Dylan and Sharon are exchanging CDs from their collections. They all have the same number of CDs. Sharon got 10 CDs from Dylan and Eric got 6 CDs from Sharon. What is the situation now? What else can you say?

**Watch out!**

In all these problems, the children are being asked to operate on relations between quantities without knowing the quantities. At the start some react by saying they don’t know how many items each of the characters in the problems had. They then decide to attribute a number to all the
characters. As long as the number is the same, their answer will be correct. However, this avoids the difficulty of operating on relations without knowing the quantities, and therefore the aim of the problem. If the children do attribute a number to the characters in order to solve the problem, ask them whether the answer would be the same if they started from a different number (e.g. if they start from 10, ask if the number would be the same if they started from 15). Usually at least one child will say that it is not necessary to attribute an arbitrary number of items to the characters. It is important to reinforce this approach.

Note also that some children mix in their answers references to quantities and to relations between quantities.

Figure 3 shows an example of a child who did start from a number and another from a child who did not.

**Figure 3.** One child attributed an arbitrary number and another did not

**Multiplicative reasoning problems will be represented using the ratio diagram**

**What do you do?**

Present the first problem on the screen and read it to the whole group: “Dazz and Mrs Elastic are doing a race around the world. Dazz takes 10 steps to go to a certain distance. Mrs Elastic takes 6 steps to go the same distance. If Mrs Elastic takes 12 steps, how many steps does Dazz have to take in order to go to the same distance?”

Allow the children to solve the problem first. Ask what their answers were.

Then introduce the diagram emphasising that what we need to find out is how many steps for the same distance. You need to emphasize that their steps are of different sizes. Dazz takes 10 steps for
every 6 steps that Mrs Elastic takes. We can then have a line for Dazz’s steps and one for Mrs Elastic and show this on the line.

Now the students are encouraged to think how many steps must Dazz take when Mrs Elastic takes 12 steps? An answer that comes up easily is that if she takes twice as many steps, he must also take twice as many steps. The diagram on the screen helps them think this through.

“*If she takes 3 steps, how many steps does Dazz have to take to cover the same distance?*” Having thought of doubling the number of steps, the children easily think of halving the number of steps. This can be discussed without showing the correspondences or these can be added to their diagrams.

Now ask the students to figure out how many steps Dazz takes when Mrs Elastic takes 15 steps. They are encouraged to think that for Mrs Elastic to go from 12 to 15 steps she has to take half of the 6 steps (which she took at the beginning) which is 3. So Dazz must also take half the number of steps.

At this initial stage in modelling multiplicative problems, you are emphasising the idea of correspondences and exploring with the children a scalar reasoning approach to solution. At a later stage, you will explore the functional reasoning in the same situations and using the same diagram.

Present the next problems and ask each pair to do a diagram and solve the problems. When they have an agreed diagram and method, they should place these on acetate and present them to the whole group.

**Problem 1:** Linda has to prepare the mix for the pancakes for her friends. The recipe says that you need to use 3 cups of flour for 8 people. She has to prepare pancakes for 4 people. How much flour does she have to use for 4 people? If she were cooking for 12 people, how much flour would she have to use?
Watch out!
The numbers in this problem are more difficult than the numbers in the previous problem, but most children in years 5 and 6 can figure out that half of 3 cups is one and a half cups. Even if they cannot write this in fractional notation, they can write it in words. They should also be able to figure out that 3 cups plus one and a half is four and a half, even if they do not know the fractional representation.

If some children get the answers quickly, ask them how much flour for 2 people. If they get stuck with solving for two people, they can be asked: “What is half of one cup? What is half of half cup?” Once again, they may be able to answer in words but not know how to carry out the computations with fractions.

While the children are presenting their solutions, you can stress that they are figuring out how much flour for the number of people.
Session 2
Start by reminding the children that they have used different types of diagrams to solve problems. They used the bar diagram and they also used the lines diagrams. They will now have to think which diagram they want to use and explain how they use the diagram to help them find the answer.

The remaining sessions will focus on problem solving and explore the use of diagrams. We have divided the problems in sessions to offer an idea of how many problems we were able to work with in the different sessions.

What do you need?
- Computer to present the problems (file titled: ‘Presentation for word problems_Unit2_Session2’, found in computer material for Unit 2)
- Children’s booklets to record their answers
- Calculators, one per pair of children – the children should be allowed to use calculators when they wish so that they focus on the relations between quantities
- Overhead projector for the children to present their solutions

What do you do?
- Present each problem on the computer screen while you are reading it.
- Prompt children to use the diagrams to solve the problems. Ask them to think whether they want to use the bar or the diagram with two lines.
- After each pair has agreed a solution to the problem, they write their agreed answer on an acetate to present to the whole group. Each pair should be asked to explain their diagram how they reached their solution.
- Ask other pairs whether they found different solutions. If this is the case, ask them to explain it to the group.
- You can have your own diagrams ready to present if you feel that more discussion of the children’s solutions is needed.

Who is taller than his brother by a bigger difference?

What do you do?
Read the following problem using the presentation on the screen to support your explanation:

Problem 1: Two friends Sam and Tim are discussing who is taller than their own brother by a bigger difference. Sam is 165 cm, Sam’s brother is 150 cm, Tim is 180 cm and Tim’s brother is 170 cm. Can you find out who is taller than his brother by a bigger difference?

The children will easily adopt the bar diagram here or draw stick figures and label them.
In this problem, they have the quantities and two differences and need to calculate the difference between the two.

Move on to the next problems and follow the same procedure: the children do their own diagrams, solve the problems in pairs, agree on a diagram and solution, write these on acetates and some are chosen to present them to the class.

**Problem 2:** George has a brother Alex who is shorter than him. Tom has a brother Dylan who is shorter than him. They were arguing who is taller by a bigger difference. George won the argument by 8 cm. George is 170 cm tall; Alex is 155 cm and Tom is 167 cm.
What can you say about the difference between Dylan and Tom?
Can you figure out how tall is Dylan?

**Problem 3:** Peter has a sister Laura who is shorter than him and Mark has a sister Debbie who is shorter than him. They argued who is taller by a bigger difference. Peter won the argument by 8 cm. Peter is 190 cm tall and Laura is 165 cm. What can you say about the difference between Debbie and Mark?

**Watch out!**
The children are used to giving answers about quantities but not about relations between quantities. Some children make a relational statement: Mark is 17 cm taller than Debbie but others find it difficult to come up with the relational statement.

**Problem 4:** Tom is 7 years old. In 4 years he will be 2 years older than Ben is now. How is Ben now?

\[
\begin{align*}
\text{7+4} & \quad \text{Ben} \\
\text{Tom} & \quad \text{Tom in 4 years} - 2
\end{align*}
\]
**Problem 5:** A recipe for strawberry muffins for 16 people shows:
- 8 cups of flour
- 2 cups of strawberries
- 8 spoons of butter
- 1 cup of sugar
- ½ cup of buttermilk

You will be cooking for 12 people. How much of each of these ingredients do you need to use?

The children do their own diagrams, solve the problems and present their solutions. At this stage, you are still exploring the scalar solutions in multiplicative reasoning problems: for half the people, half the amount. These make sense to the children but will be replaced in the diagram with functional solutions later.

**Problem 6:** A shop uses cocoa to make chocolate flakes for drinks. The shopkeeper knows that for 7.5 kg of cocoa he gets 5 kg of chocolate flakes for drinks. A lady comes in and says she wants 15 kg of chocolate flakes. How much cocoa will the shopkeeper need to use?

Once the children have solved this problem using scalar reasoning, ask them how much cocoa they need for 1 kg of chocolate flakes. They can either divide 7.5 by 5 or 22.5 by 15. If they do one calculation, ask them what number they would get if they did the other one. They should become aware that the ratio cocoa to chocolate flakes should always be the same. From this problem on, they will be presented with problems that are not easily solved using scalar reasoning and will be asked to think of the functional relation between the quantities too.

**Problem 7:** In St Peters School, there are more children in Year 3 than in Year 4. Mrs Parker bought some books for each class. Mrs Parker bought 16 books and Year 3 will get 4 books more than Year 4. How many books will each class have?
Session 3
What do you need?

- Problem cards (can be printed from the file titled: ‘Problem cards_Sessions 3-6’, found in printable resources for Unit 2)
- Children’s booklets to record their answers
- Calculators, one per pair of children – the children should be allowed to use calculators when they wish so that they focus on the relations between quantities
- Overhead projector for the children to present their solutions

Problem 1: Each pair receives a card with the problem:
In a school, students are registered to do football, ice-skating or swimming for the term. Together ice-skating and swimming have 76 students more than football has. Swimming has 32 students. Football has 48 students. How many students are there in ice-skating?

Watch out!
It is possible that students will ignore that 76 is not a quantity but a relation to the number of students in football. They should be encouraged to check their solutions in the diagram once they have finished. Figure 4 shows two children’s solutions. Note that they have checked their solutions in their diagrams by further calculations.

Problem 2: Sue and Ellie are running on a track. They run at the same speed. Sue started earlier. When Ellie started, Sue had already run 2 laps. When Ellie completes 6 laps, how many laps will Sue have completed?

(This solution can be checked by asking whether the difference in the number of laps should remain the same when they are running at the same speed or if the difference should increase.)
**Problem 3:** In Pear Tree School, there are more children in Year 5 than in Year 6. The Head Teacher bought some pizzas for each class’s lunch and the children will have to share, 2 children will share a pizza. The Head Teacher bought 16 pizzas and Year 5 is getting 4 pizzas more than Year 6. How many children are in each class?

(Not that the children should not jump at the words “2 children will share a pizza” and attempt to make this a multiplicative problem.)

![Diagram of Year 4, Year 5, and Total](image)

**Problem 4:** Eddie and Jack are cycling on the same circuit. Each of them is cycling at a constant speed but one cycles faster than the other. When Eddie had gone around the circuit 9 times, Jack had gone around the circuit 3 times. When Eddie completes the circuit 21 times, how many times will Jack have completed it?

The children will not be able to use scalar reasoning easily here. If they are finding it difficult to get started, ask them how many times would Eddie go around the circuit while Jack goes around once. You can also ask how much faster is Eddie cycling than Jack.

**Problem 5:** Ali has a sister Laura who gets less pocket money than he does and Ryan has a sister Claire who gets less pocket money than him. They argued about whether the difference between Ali’s and Laura’s pocket money was greater than the difference between Ryan’s and Claire’s. Ali won the argument: the difference between his and Laura’s pocket money was £7 greater than the difference between Ryan’s and Claire’s. Ali gets £30 and Laura gets £15. Ryan gets £40. What can you say about the difference between Ryan and Claire? Can you figure out how much pocket money does Claire get?

![Diagram of Ali, Laura, Ryan, and Claire](image)

**Problem 6:** Amy, Alex, and Julie all made some cakes for a cake sale. Alex and Julie together made 30 cakes more than Amy. Julie made 10 cakes. Amy made 14 cakes. How many cakes did Alex make?

![Diagram of Alex, Julie, and Amy](image)
Watch out!
The students are encouraged to draw the diagram for each problem and use the terms ‘Total’ and ‘Difference’ in their diagrams. It is possible that students will find these two problems quite challenging but the diagrams will assist them to tackle the solution.

Problem 7: Ellen bought 8 helium-filled balloons and paid £24. She went back and bought 16 balloons for her class. How much did she have to pay for 16 balloons?

Students are encouraged to draw a diagram that shows how much she paid for 8 balloons. The ratio diagram will facilitate the discussion of the cost of 16 balloons as a scalar solution. They can then explore the diagram by finding out the price for 4, 12 and 1 balloon. Ask how they found the price for 1 balloon and whether the price would be the same if they carried out a different division (for example, if they divided 12 by 4, ask about 48 by 16). They are encouraged to look at the pattern on the diagram (48:16 = 3, 36:12 = 3, 24:8 = 3, 12:4 = 3, 3:1 = 3).

The aim is to make them aware that there is a ratio between pounds and balloons which is always the same. The children would build a ratio table like the one below and discuss that the ratio is always 1 balloon for 3 pounds.

![Ratio Table for Problem 7]

Problem 8: Granny has to put fertilizer on her plants in her garden. The instruction says that she has to mix 2 measures of fertilizer with 8 liters of water (the measure comes in the box). She has a tank with 24 liters of water in it. How many measures of fertilizer does she have to put into her tank?

When the ratio table is built, the students can explore the scalar and functional relations. If she uses 16 liters of water, how much fertilizer does she need?
Session 4

What do you need?

- Problem cards (can be printed from the file titled: ‘Problem cards_Sessions 3-6’, found in printable resources for Unit 2)
- Children’s booklets to record their answers
- Calculators, one per pair of children – the children should be allowed to use calculators when they wish so that they focus on the relations between quantities
- Overhead projector for the children to present their solutions

Problem 1: Peter, Tom and Jane combined their apples for a fruit stand. Tom and Jane together has 97 apples more than Peter. Jane had 17 apples. Peter has 25 apples. How many apples did Tom have?

The children will need to use a bar graph. In order to allow for different rates of work, further problems can be presented on cards, which the children take when they have finished the previous problem. The teacher will need to monitor progress and bring the class together for discussion of some of the problems. Students who finish more quickly can be asked to explain how they found the answer.

Problem 2: Each pair receives a card with the problem presented in a table.

Jim has to print the school newspaper but he can only do it in break time. It takes him 15 minutes to print 12 newspapers. How many newspapers can he print during the 35 minutes break?
**Problem 3:** To feed 3 rabbits you need 270 grams of rabbit food a week. The food comes only in bags of 540 grams. How many bags do you need to feed 15 rabbits in a week?

![Diagram of rabbits and grams]

**Watch out!**
It is possible that different students will solve the problem in different ways. Some might look at the number of bags for 3 rabbits (half a bag a week for 3 rabbits) and then multiply half by five. Others might start by working out how many grams 15 rabbits will eat in a week.

**Problem 4:** The Year 4 won 3 prizes more than the Year 5 and the Year 6 won 5 prizes more than the Year 4. The school gave out 68 prizes. How many prizes did the champion class win?

Figure 5 shows some examples of students’ solutions below, which illustrate two difficulties.

![Figure 5: Children’s solutions of the current problem]

The student whose work appears on the left is still using a hypothetical number and then adjusting the totals, a method that she gets to work in this problem but is rather awkward. She initially rejected division as a relevant operation because the classes had not received the same number of prizes. This can be a point for discussion when the solutions are presented. The student whose work appears on the right recognised the relevance of division, but did not add the relations appropriately: Year 6 received 5 more than Year 4 and therefore 8 more than Year 5. Before the students discuss their solutions, the teacher can ask which class won most prizes, which won the least, discuss the differences between the classes, and then move on to the students’ presentations. This would help some students realise that they skipped one step in solving the problem.
**Problem 5:** Sam is 27 years older than Toby and Toby is 13 years younger than Chloe. How much older is Sam than Chloe?

![Diagram](image)

**Watch out!**
In order to come up with the solution, it is possible that students will find it easier to give numbers to represent the age of each person. Ask the students if they need to know the age of each person (exact number) in order to tackle the solution. Students should be encouraged to think about the difference in years between the 3 people.

**Problem 6:** In a school there is a hamster. It eats 10.5 scoops of food a week. Clara is looking after it for 14 days. How many scoops does she need? Jane is looking after the hamster for 18 days. How many scoops will she need?

The problem is read to the children and each pair is given a card with the drawings

**Problem 7:** On the card there are drawings of a wide and a narrow cylinder. The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the 4th mark (see A). This water reaches the 6th mark when poured into the narrow cylinder (see B).

Both cylinders are emptied and then some water is poured into the wide cylinder up to the 7th mark. How high would this water rise if it were poured into the narrow cylinder?
Watch out!
This problem attracts many additive reasoning solutions. The children draw the line diagram but think that if the water is 2 marks higher in the wide cylinder, it will also be 2 marks higher in the narrow one. Functional reasoning appears easier when the students can give the ratio a meaning, such as amount of food per day, price per balloon, time needed to print one newspaper etc. This problem can help students realise why it is important to ask the students to check their answers by making other comparisons using the diagram. The students can be asked if the water is up to 2 in the wider cylinder, which is half of 4, where will it be in the narrower one? And if it is up to the mark 1 in the wider cylinder, where will it be in the narrower one? Figure 6 shows one student’s solution when this checking was carried out. The students are then asked to think about which of the solutions is better and why.
Session 5

- Problem cards (can be printed from the file titled: ‘Problem cards_Sessions 3-6’, found in printable resources for Unit 2)
- Children’s booklets to record their answers
- Calculators, one per pair of children – the children should be allowed to use calculators when they wish so that they focus on the relations between quantities
- Overhead projector for the children to present their solutions

Problem 1

Present the map with the different temperature reported on Monday and that one with the temperature reported on Tuesday. Ask the children to look at both maps and then answer the following questions in the booklets:

![Maps showing temperature differences](image)

Question: What is the temperature of Rio on Monday?
Question: What is the temperature of Athens on Tuesday?
Question: What's the difference of temperature between Rio and Athens on Tuesday?
Question: What happened in the night at Moscow?
Question: What's the difference of temperature between Moscow and Rio on Tuesday?

Watch out

It is expected that students will find the first two questions quite easy to tackle but they may encounter difficulties with the last two questions. It is possible that when attempting to solve the last question they would use a quantity instead of considering the relation between the quantities.
- Each pair receives a card with the problem

**Problem 2:** In the bathroom, Paul replaced 5 energy saving bulbs and he saved £20 that year. Then, he decided to replace all the bulbs in the kitchen as well. There are 8 bulbs in the kitchen. How much money does he save?

**Problem 3:** James, Kelly and Donna were playing a computer game. At the beginning they have the same number of points. James played against Kelly and he lost 20 points which went to Kelly. Then, Kelly played against Donna and Kelly lost 30 points which went to Donna.

What is the difference in points between James and Kelly now? What is the difference in points between Donna and Kelly? What is the difference in points between Donna and James?

**Watch out!**
It is possible that students might want to use a quantity at the beginning in order to solve the problem. It is important to allow students to use their own strategies but also to point out the efficiency of working on the relations without making up an initial amount.

**Problem 4:** Helen had some sweets. She gave half to her friend James and 3 to her cousin Alex. Her grandfather gave her 20 more the next day. At the end she had 23 sweets left.

How many sweets did she have at the beginning?

**Watch out!**
Some students might not realise that “half” means divided by 2 and then not know how to use multiplication as the inverse of division.
Session 6

- Problem cards (can be printed from the file titled: ‘Problem cards_Sessions 3-6’, found in printable resources for Unit 2)
- Children’s booklets to record their answers
- Calculators, one per pair of children – the children should be allowed to use calculators when they wish so that they focus on the relations between quantities
- Overhead projector for the children to present their solutions

**Problem 1:** You can earn £84 in 8 weeks but you can only work for 5 weeks because you are going on a school trip. How much will you earn in 5 weeks?

The problem is presented as a pay slip.

**Watch out!**
The students should be encouraged to use the functional relation and speak about amount of money per week.

**Problem 2:** You are on A34 towards Banbury. Banbury is 30 miles from Oxford and Bicester is 18 miles from Oxford. We know that from Oxford to Banbury takes about 40 minutes. How long does it take from Oxford to Bicester?

**Problem 3:** Mrs. Parker had collected some school vouchers. A parent gave her 22 vouchers. She spent a quarter of her vouchers to buy books and 40 to buy sports equipment. At the end she did not have any left. How many vouchers did she have at the beginning?

**Problem 4:**

What is the time difference between London and Tonga?
What is the time difference between London and Mexico City?
What is the time difference between Tokyo and Washington?

**Watch out**
Students may need time to study this map so explore it with them before they start. They may need some prompting to think about midnight as zero hours.
Appendix A

Computer Game Level 1 Questions

1. Tom has some keys in a wallet. Mike gives Tom 2 keys to the new lockers. Now Tom has 8 keys. How many keys did Tom have before Mike gave him the new keys?

2. Jack has 8 jars of honey. He goes to the school fete and sells 3 jars of honey. How many jars does Jack have left in his cupboard after the fete?

3. Jack is goalkeeper for the team. In the second half, he lets in 2 goals. Now the away team has 6 goals. How many goals did the away team score in the first half?

4. Harry is having a party, so he makes 2 pizzas. Before the guests arrive he decides that he needs more food so he makes 8 more pizzas. How many pizzas did Doug make altogether?

5. Owen has a bag of cookies. A Fox takes 3 cookies from the bag. Owen now has 4 cookies left. How many cookies did Owen have in the bag?

6. Adam bought some tomatoes for the class picnic. Sam calls and gives him 4 more tomatoes. Now Adam has 7 tomatoes. How many tomatoes did Adam have for the picnic before Sam called round?

7. Josie has 4 stamps in her new book. Ben gives her 2 more. How many stamps does Josie have now?

8. The boys are playing with 5 beach balls. When Charlie arrives, he brings 4 balls with him. How many balls do they have in the bag to play with?

9. Jack has 8 bottles of water for sports day. After his first race, he drinks 2 of the bottles. How many drinks does Jack have left in his backpack?

10. Laura has some CD’s. When Mike visits, Laura gives him 6 CD’s. Now Laura has 2 left. How many CD’s did Laura have to start with?
Computer Game Level 2 Questions

1. Bella is waiting for granny to come to tea, she has 2 cakes. Granny brings 3 cakes for them to Bella’s house. How many cakes do they have altogether in the cake tin?

2. Mum is tidying her jewel box. She has some bracelets, she leaves 9 of them in the box and one she takes out to wear. How many bracelets were in the box in the beginning?

3. Ricky picked some apples from the tree. He gives 4 apples to Tom for helping him with the gardening. Now he has 5 apples left. How many apples did Ricky pick from the tree?

4. Suzy is playing with 5 friends. 4 of them have to go home for their lunch. How many friends will be left for Suzy to play games with?

5. Thomas has a pond with some frogs in the bottom. Dad brings him 2 frogs for his pond. Now Thomas has 10 frogs jumping in his pond. How many frogs were in the pond before Dad brought him the frogs?

6. Emma has 3 sweets which she takes out of her bag. Her brother gives her 4 more. How many sweets does Emma have in her bag now?

7. The cat sees 7 mice in the house. Then it sees 3 mice in the garden. How many mice did the cat see on its walk before it went to sleep?

8. It’s Katherine’s birthday and she has some presents. When Ted arrives he brings Katherine 2 more presents. Altogether Katherine received 5 presents. How many presents did she receive before Ted arrived?

9. Mum makes 8 strawberry milkshakes. Rosie drinks 2 milkshakes. How many milkshakes does Mum have left in her box?

10. Grace received some flowers from the delivery man. She put 4 of them in a pot. Then there were 5 flowers left. How many flowers did she have before she put some in a pot?
1. In the garden, there are some apple trees. 19 apples fell off the trees. The other 16 were still on the trees. How many apples were on the trees before the 19 fell off?

2. Alex has 18 boxes of strawberries. He buys 26 more boxes. How many boxes of strawberries does Alex now have?

3. Peter sang in a concert. 12 people left after the break. At the end there were still 27 people. How many people were in the theatre at the beginning?

4. Harry puts 15 cookies in his bag. Martha bakes 13 cookies for him. How many cookies does Harry have now?

5. Sophie has some tickets for the school dance. Toby gave her 17 tickets. Now Sophie has 25 tickets. How many tickets does Sophie have at the beginning?

6. Jack has 18 sweets in his bag. Owen gives him 22 sweets more. How many sweets were in the bag?

7. Peter is planting some new plants for his garden. George brings Peter 12 more plants. Now Peter has 35 plants for his garden. How many plants did Peter have before he saw George?

8. Chloe has received some flowers by delivery for her birthday. Her friend brought her 13 more flowers. Now Chloe has 29 flowers in her house. How many flowers did Chloe receive before her friends brought her some more flowers?

9. Zak found 19 crayons to use for colouring his graph. His teacher gave him 18 more crayons. How many crayons did Zak have?

10. The postman has 43 letters to deliver. Two dogs took a bag with 25 letters in it. How many letters are left in the postman’s bag?
1. Rachel has a bag full of books for the school fete. Sarah brings her 14 more books. Now Rachel has 35 books. How many books did Rachel have in her bag before Sarah brought some?

2. Freya and Theo pick some flowers from the field behind their house. They put 16 flowers in a vase for their mum. 13 flowers are left. How many flowers were picked?

3. Joe and Julie are getting married. They bought some bottles of drink for the party. Penny brings them 17 bottles. Now Joe and Julie have 32 bottles for the party. How many bottles did they have before Penny brought them some more?

4. Luke has a box of 52 sweets to take to share with his friends. He gives 26 to his friends. How many sweets are left in Luke’s box?

5. Some hens laid some eggs in their nests. A fox takes 29 of them. Now the hens have 17 eggs left. How many eggs did the hens lay?

6. Spike likes to keep bones in the house. He decides to bury 15 of his bones in the garden. Now there are 8 bones left in the house. How many bones were in Spike’s house before he buried some?

7. Dan likes collecting caterpillars. He has lots of caterpillars in his collection. 32 caterpillars crawled away. Now there are only 14 caterpillars left in his collection. How many were in his collection before some crawled away?

8. Carl has some cans of drink in the boot of his car. He gives 18 cans to Harry. Now he has 13 cans left. How many cans did Carl have before he gave some away?

9. Grand dad grew some carrots in his garden and put them in a box. When Bill called to help in the garden, Grand dad gave him 29 carrots to take home. Grand dad now has 17 carrots left. How many carrots did Grand dad have in his box before Bill came?

10. Brian has a bag with 26 marbles in it. Percy gives Brian 15 marbles. How many marbles does Brian have now?